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A NEW MATHEMATICAL METHOD FOR DESCRIPTION OF DYNAMICS OF SPINNING PARTICLES IN GENERAL RELATIVITY: GRAVITOELECTROMAGNETISM

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Abstract

The new conception of gravitoelectromagnetism based on the Pomeransky-Khriplovich gravitoelectromagnetic fields and the Schwinger gauge is expounded and compared with the conventional one. Unlike the conventional conception, the new one is applicable for relativistic spinning particles in arbitrarily strong gravitational fields and perfectly describes the evolution of momentum and spin. This conception presents a new approach for description of dynamics of spinning particles in general relativity which applies equations of motion in an appropriate anholonomic frame. The main distinctive feature of the gravitoelectromagnetic fields is their dependence on the particle four-momentum. The simple explicit expressions for the gravitoelectromagnetic fields in the weak-field approximation allow to explain their general properties. The gravitoelectromagnetic fields are calculated for the mainly used space-times. The new conception of gravitoelectromagnetism is applied for deriving the simple expression for the Mathisson force in the weak-field approximation.

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1 Introduction

Great success achieved in description of electromagnetic phenomena stimulated a search for direct analogies between the electrodynamics and gravity. Indeed, the Newton and Coulomb laws as well as the Coriolis and Lorentz forces seems to be similar. This is the reason why the conventional conception of gravitoelectromagnetism is based on the introduction of scalar and vector potentials of gravitoelectromagnetic field. Then one introduces the gravitoelectic and gravitomagnetic fields connected each with others by Maxwell-like equations (see Refs. [1, 2, 3] and references therein). Since mathematical tools of the electrodynamics and gravity significantly differ, this conception allows only approximate description of gravitational phenomena.

New big step in formulation of more exact equations of gravitoelectromagnetic field has been made by Pomeransky and Khriplovich [4]. They have started from standard equations stating the equality of covariant derivatives of the four-spin and four-velocity to zero. Under these conditions, a transition to a local Lorentz frame allows to derive general equations for the tetrad components of the four-spin and four-velocity. The obtained equations are pretty similar to the corresponding equations in electrodynamics: the Thomas-Bargmann-Michel-Telegdi equation (for a particle with the Dirac magnetic moment) and the covariant equation of motion of a charged particle. This similarity allows to derive general formulas for the gravitoelectromagnetic fields [4] valid for a *relativistic* particle in an *arbitrarily strong* gravitational field. The Pomeransky-Khriplovich (PK) method defines these fields in local Lorentz (tetrad) frames. It is important to mention that this method is based on the equivalence principle extended on the spin by Kobzarev and Okun [5] (see also Ref. [6]).

The most general description of motion of spinning particle in general relativity is provided by the Mathisson-Papapetrou (MP) equations [7, 8]. These equations predict the violation of the weak equivalence principle for point-like spinning particles [9]. Nevertheless, the MP and PK equations agree when one can neglect the mutual influence of particle and spin motion leading to the above violation [10]. This circumstance substantiates the results obtained by Pomeransky and Khriplovich and brings a possibility of

a wide application of new approach to the gravitoelectromagnetism.

Set of previously obtained results allows us to formulate the conception of gravitoelectromagnetism based on the PK gravitoelectromagnetic fields and the Schwinger gauge. PK equations define the general form of the gravitoelectromagnetic fields. However, the weak-field approximation happens to be very convenient to obtain simple expressions for these fields [10] clarifying their physical meaning. We obtain explicit formulas for the gravitoelectromagnetic fields corresponding some generally used spacetimes just in this approximation. We exactly calculate the gravitoelectromagnetic fields in noninertial frames to show their main distinctive features. We also analyze the origin of the Mathisson force and demonstrate its similarity to the Stern-Gerlach force in electrodynamics.

The paper is organized as follows. In Sec. 2, we describe the conventional conception of gravitoelectromagnetism and give examples of gravitoelectromagnetic fields obtained with this conception. Short discussion of the MP equations and an approximation in which the equivalence principle (extended on the spin) is restored is given in Sec. 3. We introduce the PK equations in Sec. 4 and find an explicit dependence of gravitoelectromagnetic fields and equations of motion on a gauge in Sec. 5. Gravitoelectromagnetic fields for specific spacetimes are calculated in Sec. 6. The expression for the Mathisson force in terms of these fields is derived in Sec. 7. The results obtained are summarized in Sec. 8.

We denote world and spatial indices by Greek and Latin letters $\alpha, \mu, \nu, \ldots = 0, 1, 2, 3$, $i, j, k, \ldots = 1, 2, 3$, respectively. Tetrad indices are denoted by Latin letters from the beginning of the alphabet, $a, b, c, \ldots = 0, 1, 2, 3$. Temporal and spatial tetrad indices are distinguished by hats. The signature is $(+---)$, the Ricci scalar curvature is defined by $R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu}$, where $R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}{}_{\mu\nu} - \ldots$ is the Riemann curvature tensor.

2 Conventional conception of gravitoelectromagnetism

The conventional conception of gravitoelectromagnetism (see Refs. [1, 2, 3]) is based on a four-potential of gravitoelectromagnetic field expressed in

terms of components of the metric tensor. This conception can be applied in the nonrelativistic and weak-field approximation. The scalar (Φ) and vector ($\boldsymbol{A}$) potentials of the gravitoelectromagnetic field are defined by

$$\Phi = \frac{c}{2}\left(g_{00} - 1\right), \quad A^i = -\frac{c}{2}\delta^{ij} g_{0j}. \tag{1}$$

They satisfy the gauge condition

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \tag{2}$$

The gravitoelectromagnetic field strengths and the gravitational force are given by

$$\mathfrak{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = 0, \quad \mathfrak{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{F} = mc\mathfrak{E} + m\boldsymbol{v} \times \mathfrak{B}, \tag{3}$$

where m is the mass of the particle.

Let us calculate the gravitoelectromagnetic fields for some important spacetimes. The general form of the line element of an arbitrary gravitational field is given by [11, 12]

$$ds^2 = V^2 c^2 dt^2 - \delta_{\widehat{ij}} W^{\widehat{i}}{}_k W^{\widehat{j}}{}_l \left(dx^k - K^k c dt\right)\left(dx^l - K^l c dt\right). \tag{4}$$

An analysis of main gravitational fields can be simplified with the use of isotropic coordinates. In this case, the line element takes the form (see Ref. [13])

$$ds^2 = V^2 c^2 dt^2 - W^2\,\delta_{ij}\left(dx^i - K^i c dt\right)\left(dx^j - K^j c dt\right). \tag{5}$$

The general case of inertial fields in a flat spacetime is presented by an accelerated and rotating noninertial frame. The *exact* metric of this frame has also form (5), where [14]

$$V = 1 + \frac{\boldsymbol{a} \cdot \boldsymbol{r}}{c^2}, \quad W = 1, \quad \boldsymbol{K} = -\frac{1}{c}\left(\boldsymbol{\omega} \times \boldsymbol{r}\right). \tag{6}$$

Here $\boldsymbol{a}$ is an acceleration of the observer and $\boldsymbol{\omega}$ is an angular velocity of a noninertial reference system. Both of them are independent of the spatial coordinates, but may depend arbitrarily on time.

In this case, the gravitoelectromagnetic fields (in the weak-field approximation) are equal to

$$\mathfrak{E} = -\frac{\boldsymbol{a}}{c}, \quad \mathfrak{B} = \boldsymbol{\omega}. \tag{7}$$

Another important example is the Lense-Thirring (LT) metric [15] defining the gravitational field of a rotating source in the weak-field approximation. Its static part characterizes the Schwarzschild field when a distance from the source is large in comparison with the gravitational radius. It is convenient to transform the LT metric to the isotropic coordinates [13]:

$$\begin{gathered} V = 1 - \frac{GM}{c^2 r}, \quad W = 1 + \frac{GM}{c^2 r}, \quad \boldsymbol{K} = \frac{\boldsymbol{\omega} \times \boldsymbol{r}}{c}, \\ \boldsymbol{\omega} = \frac{2G}{c^2 r^3}\boldsymbol{J} = \left(0,\ 0,\ \frac{2GMa}{c\, r^3}\right). \end{gathered} \tag{8}$$

Here $\boldsymbol{J} = Mca\boldsymbol{e}_z$ and M are the total angular momentum and the mass of the source. This metric can also be presented in the form

$$ds^2 = \left(1 + \frac{2\Phi}{c}\right) c^2 dt^2 - 4\boldsymbol{A} \cdot d\boldsymbol{r}\, dt - \left(1 - \frac{2\Phi}{c}\right) \delta_{ij} dx^i dx^j, \tag{9}$$

where

$$\Phi = -\frac{GM}{cr}, \quad \boldsymbol{A} = -\frac{c}{2}\boldsymbol{K} = -\frac{G}{c^2 r^3}\boldsymbol{J} \times \boldsymbol{r}. \tag{10}$$

The gravitoelectromagnetic fields in the LT spacetime are given by

$$\mathfrak{E} = -\frac{GM\boldsymbol{r}}{cr^3}, \quad \mathfrak{B} = -\frac{G}{c^2 r^3}\left[\frac{3(\boldsymbol{r} \cdot \boldsymbol{J})\boldsymbol{r}}{r^2} - \boldsymbol{J}\right]. \tag{11}$$

The static metric in the isotropic coordinates is defined by Eq. (5) at condition that $\boldsymbol{K} = 0$. In this case, the gravitoelectromagnetic fields read

$$\mathfrak{E} = -c\nabla V, \quad \mathfrak{B} = 0. \tag{12}$$

In the more general case of arbitrary V, W, and $\boldsymbol{K}$ in Eq. (5), the gravitoelectric field remains the same and the gravitomagnetic field takes the form

$$\mathfrak{B} = -\frac{c}{2}\nabla \times \boldsymbol{K}. \tag{13}$$

It is important that the gravitoelectromagnetic fields *do not depend* on W.

The conventional conception of the gravitoelectromagnetism is pretty simple and main definitions are similar to those in electrodynamics. However, the potentials Φ and $\boldsymbol{A}$ can be used only in the nonrelativistic and weak-field approximation.

3 Mathisson-Papapetrou equations

To elaborate the more general conception of the gravitoelectromagnetism, we need to use an approach based on the famous MP equations. These equations were first found by Mathisson [7] and then rediscovered by Papapetrou [8]. They give a classical description of dynamics of the momentum and spin of a spinning particle in curved spacetimes. All multipole moments of an extended body in a gravitational field have been taken into account by Dixon [16]. The explicit form of the MP equations is

$$\frac{Dp^{\mu}}{ds} = -\frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}u^{\nu}S^{\alpha\beta}, \tag{14}$$

$$\frac{DS^{\mu\nu}}{ds} = p^{\mu}u^{\nu} - p^{\nu}u^{\mu}, \tag{15}$$

where u^{ν} and p^{ν} are the four-velocity and four-momentum of the spinning particle, respectively, $R^{\mu}_{\nu\alpha\beta}$ is the Riemann curvature tensor of the spacetime, and $D/(ds)$ means the covariant derivative with respect to the interval ds.

These equations should be supplemented with the condition [7, 17]

$$S^{\mu\nu}u_{\nu} = 0 \tag{16}$$

or [16, 18, 19]

$$S^{\mu\nu}p_{\nu} = 0. \tag{17}$$

The MP equations characterize the pole-dipole approximation, when multipole moments of higher orders are neglected. These equations predict the mutual influence of particle and spin motion. In particular, spinning particles undergo an additional force which is similar to the Stern-Gerlach

force in electrodynamics. As a result, spinning particles do not move on geodesics in curved spacetimes.

In a zero approximation, one can neglect the mutual influence of particle and spin motion. In this approximation, the spin tensor is parallel transported in the spacetime and the MP equations take the form

$$\frac{Dp^\mu}{ds} = 0, \tag{18}$$

$$\frac{DS^{\mu\nu}}{ds} = 0, \tag{19}$$

$$p^\mu = mcu^\mu, \tag{20}$$

Eqs. (18) and (18) formulate the conventional equivalence principle and its extension on the spin, respectively. When these equations are satisfied, particles move on geodesic lines and all spins rotate with the same angular velocity.

To consider an availability of Eq. (19), we use the results obtained by Chicone, Mashhoon, and Punsly [20]. The relation between the four-momentum and four-velocity has the form

$$p^\mu = mcu^\mu + E^\mu, \tag{21}$$

where the order of magnitude of E^μ is given by

$$E^\mu \sim \frac{1}{mc} S^{\mu\nu} \frac{Dp_\nu}{ds}. \tag{22}$$

The additional four-force is $DE^\mu/d\tau$, where τ is the proper time. This four-force is of the second order in the spin [20]. The approximate equation of the first order in the spin resulting from Eq. (14) is [20]

$$mc\frac{Du^\mu}{ds} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta}. \tag{23}$$

This equation describes the effect of the Mathisson force on the particle motion. Evidently, one need not calculate the correction to the spin motion equation (19) in order to determine this force. This circumstance will be used in Sec. 7.

Equations (14),(21),(22) and the PK equations unambiguously show that the spin dynamics depends on derivatives of the metric tensor. The right hand side of Eq. (15) defined by Eqs. (14),(21), and (22) is of order of

$$p^\mu u^\nu - p^\nu u^\mu \sim \frac{1}{mc} R_{\lambda\alpha\beta\gamma} u^\alpha u^\nu S^{\mu\lambda} S^{\beta\gamma}. \tag{24}$$

This quantity is much less than terms defining the spin motion in the PK equations, when the weak-field approximation is used. In this approximation

$$|g_{\mu\nu} - \eta_{\mu\nu}| = |h_{\mu\nu}| \ll 1, \tag{25}$$

where the tensor $\eta_{\mu\nu}$ characterizes the Minkowski spacetime. In addition, the right hand side of Eq. (24) is of the second order in the spin tensor. Therefore, the correction to Eq. (19) is rather small.

When only terms of the first order in the spin are retained, the MP equations reduce to Eqs. (19) and (23). In this case, the corresponding equation for the spin (pseudo)vector S^μ is given by

$$\frac{DS^\mu}{ds} = 0. \tag{26}$$

Such a situation takes place in the quantum mechanics of Dirac particles [21] because any terms bilinear in the spin matrices can always be simplified: $\Sigma^a \Sigma^b = \delta^{ab} + i\epsilon^{abc}\Sigma^c$. As a result, the equation of spin motion of a Dirac particle does not contain such terms.

Eq. (26) should be supplemented with the orthogonality condition

$$S^\mu u_\mu = 0. \tag{27}$$

4 Pomeransky-Khriplovich equations

The general conception of the gravitoelectromagnetism allowing for an introduction of the gravitoelectromagnetic fields has been proposed by Pomeransky and Khriplovich [4]. Thereinafter, commas and semicolons denote partial and covariant derivatives, respectively. When Eq. (19) is satisfied,

$$DS^a = D(S^\mu e^a_\mu) = S^\mu e^a_{\mu;\nu} dx^\nu. \tag{28}$$

Therefore,

$$\frac{DS^a}{d\tau} = S^\mu e^a_{\mu;\nu} u^\nu = S^\gamma e^c_\gamma e^\mu_c e^a_{\mu;\nu} e^\nu_d e^d_\lambda u^\lambda = \eta^{ab} S^c \Gamma_{bcd} u^d, \tag{29}$$

where $\eta^{ab} = diag\{1, -1, -1, -1\}$ defines the Minkovski metric and $\Gamma_{abc} = -\Gamma_{bac} = e^\mu_b e^\nu_c e_{a\mu;\nu}$ are the Ricci rotation coefficients. They can also be presented in the form

$$\Gamma_{abc} = \frac{1}{2}\left(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}\right), \quad \lambda_{abc} = -\lambda_{acb} = e^\mu_b e^\nu_c (e_{a\mu,\nu} - e_{a\nu,\mu}). \tag{30}$$

Since $S^a = S^\mu e^a_\mu$ is a scalar, $DS^a = dS^a$ and Eq. (29) takes the form [4]

$$\frac{dS^a}{d\tau} = \eta^{ab} \Gamma_{bcd} S^c u^d. \tag{31}$$

Equation (31) defines the spin precession which angular velocity is given by [4]

$$\Omega_{\hat{i}} = c e_{ikl} \left(\frac{1}{2} \Gamma_{klc} + \frac{u^{\hat{k}}}{u^{\hat{0}} + 1} \Gamma_{0lc} \right) \frac{u^c}{u^0}. \tag{32}$$

When the Mathisson force is not taken into account, the equation for the four-velocity is similar to Eq. (31) [4]:

$$\frac{du^a}{d\tau} = \eta^{ab} \Gamma_{bcd} u^c u^d. \tag{33}$$

Evidently, Eq. (31) is similar to Thomas-Bargmann-Michele-Telegdi equation describing the spin motion of Dirac particles in an electromagnetic field:

$$\frac{dS_\mu}{d\tau} = \frac{e}{m} F_{\mu\nu} S^\nu. \tag{34}$$

Equation (33) has the same form as the Lorentz equation:

$$\frac{du_\mu}{d\tau} = \frac{e}{m} F_{\mu\nu} u^\nu. \tag{35}$$

The tensor of electromagnetic field can be expressed in terms of the electric and magnetic fields: $F_{\mu\nu} = (\boldsymbol{E}, \boldsymbol{B})$. A similarity between $\frac{e}{m} F_{\mu\nu}$ and $\Gamma_{abc} u^c$

allows to introduce the gravitoelectric and gravitomagnetic fields: $\Gamma_{abc}u^c = (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}})$. Explicitly [4, 12]

$$\mathcal{E}_{\hat{i}} = c\Gamma_{0ic}u^c, \quad \mathcal{B}_{\hat{i}} = -\frac{c}{2}e_{ikl}\Gamma_{klc}u^c. \tag{36}$$

For the gravitoelectromagnetic fields, we do not make a difference between upper and lower indices. The equations of motion expressed in terms of these fields take the form

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{s}, \quad \boldsymbol{\Omega} = \frac{1}{u^0}\left[-\boldsymbol{\mathcal{B}} + \frac{\widehat{\boldsymbol{u}} \times \boldsymbol{\mathcal{E}}}{u^{\hat{0}} + 1}\right], \tag{37}$$

$$\frac{d\widehat{\boldsymbol{u}}}{dt} = \frac{u^{\hat{0}}}{u^0}\left(\boldsymbol{\mathcal{E}} + \frac{\widehat{\boldsymbol{u}} \times \boldsymbol{\mathcal{B}}}{u^{\hat{0}}}\right), \quad \frac{du^{\hat{0}}}{dt} = \frac{\boldsymbol{\mathcal{E}} \cdot \widehat{\boldsymbol{u}}}{u^0}. \tag{38}$$

Simplifying the description of motion of the spinning particle in gravitational fields and noninertial frames is the most important preference of introduction of the gravitoelectromagnetic fields.

A comparison of main equations of motion demonstrates that a deep similarity of gravity and electromagnetism exists even in the relativistic case. However, there is an important specific feature of the gravitoelectromagnetic fields. This is their dependence on the particle four-velocity. It is also important that the gravitoelectromagnetic fields are defined in local Lorentz (tetrad) frames.

5 Explicit dependence of gravitoelectromagnetic fields and equations of motion on a gauge

The derivation of Eq. (32) in Ref. [4] is based on the analogy between the motion of the spin in gravitational and electromagnetic fields expressed by Eqs. (31) and (34). However, the equation of spin motion in the electromagnetic field was derived for the Minkowski metric. Therefore, the physical meaning of Eq. (32) is to be carefully checked.

It has been explicitly shown in Ref. [10] (see also Ref. [13]) that the equations of motion significantly depend on a choice of gauge. In particular, the symmetric gauge used in Ref. [4] is at least useless for a description of gravitoelectromagnetic effecs in nonstatic spacetimes. We fulfill the detailed analysis of this problem and obtain an explicit dependence of gravitoelectromagnetic fields on the gauge in the weak-field approximation.

In this approximation,

$$g_{0i} = e_{\hat{0}i} + e_{\hat{i}0}, \quad \lambda_{abc} = e_{\hat{a}b,c} - e_{\hat{a}c,b}. \tag{39}$$

In the right hand side of the latter relation, the indices without hats relate to world coordinates.

For any natural gauge, $e_0^{\hat{0}}$ and $e_j^{\hat{i}}$ do not depend on g_{0i}. Therefore,

$$e_{\hat{0}0} = 1 + \frac{1}{2}h_{00}, \quad e_{\hat{i}j} = -\delta_{ij} + \frac{1}{2}h_{ij}. \tag{40}$$

The nonzero Ricci rotation coefficients are given by [10]

$$\begin{gathered}\Gamma_{\hat{i}\hat{0}\hat{0}} = -\Gamma_{\hat{0}\hat{i}\hat{0}} = \frac{1}{2}g_{00,i}, \quad \Gamma_{\hat{i}\hat{0}\hat{j}} = -\Gamma_{\hat{0}\hat{i}\hat{j}} = \frac{1}{2}\left(e_{\hat{i}0,j} - e_{\hat{0}i,j} + g_{0j,i}\right), \\ \Gamma_{\hat{i}\hat{j}\hat{0}} = \frac{1}{2}\left(g_{0j,i} - g_{0i,j}\right), \quad \Gamma_{\hat{i}\hat{j}\hat{k}} = \frac{1}{2}\left(g_{jk,i} - g_{ik,j}\right).\end{gathered} \tag{41}$$

As a result, the gravitoelectromagnetic fields are equal to [10]

$$\begin{aligned}\mathcal{E}_{\hat{i}} &= -\frac{c}{2}\left[g_{00,i}u^{\hat{0}} + \left(e_{\hat{i}0,j} - e_{\hat{0}i,j} + g_{0j,i}\right)u^{\hat{j}}\right], \\ \mathcal{B}_{\hat{i}} &= \frac{c}{4}e_{ijk}\left[\left(g_{0j,k} - g_{0k,j}\right)u^{\hat{0}} + \left(g_{jl,k} - g_{kl,j}\right)u^{\hat{l}}\right].\end{aligned} \tag{42}$$

Evidently, the gravitoelectric field depends and the gravitomagnetic field does not depend on the gauge. For the gravitoelectromagnetic fields, we do not make a difference between the upper and lower indices.

All effects of the general relativity can be properly described with any tetrad. Nevertheless, different tetrads correspond to different reference frames.

The connection between tetrad and world coordinates

$$dx^a = e_\mu^a dx^\mu, \quad dx'^a = e'^a_\mu dx^\mu \tag{43}$$

leads to the relationship between different tetrad coordinates:

$$dx^a = T^a_b dx'^b, \quad T^a_b = e^a_\mu e'^\mu_b. \tag{44}$$

The connection between the the tetrad components of the four-velocity u^μ has the form

$$u^a \equiv \frac{dx^a}{ds} = T^a_b u'^b. \tag{45}$$

Equations (43) and (45) show that the tetrad components of the four-velocity u^μ describe the four-velocity of a test particle in corresponding frames. The gauge significantly affects the velocity of the test particle in a chosen tetrad frame. Equations (44) and (45) express the Lorentz transformation between two inertial frames. The primed frame moves relative to the unprimed one with the velocity $\boldsymbol{V}$ [10]. In the weak field approximation, $|g_{0i}| \ll 1$, $V/c \ll 1$, and Eq. (45) takes the form

$$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}' + \boldsymbol{\beta} u^{\widehat{0}}, \quad u^{\widehat{0}} = u'^{\widehat{0}} + \boldsymbol{\beta} \cdot \widehat{\boldsymbol{u}}, \tag{46}$$

where $\boldsymbol{\beta} = \boldsymbol{V}/c$. When the quantity $\boldsymbol{V}$ depends on time, the relative motion of two frames is accelerated.

In the weak-field approximation,

$$\begin{gathered} g_{0i} = g^{0i}, \quad g_{0i} = e^{\widehat{0}}_i - e^{\widehat{i}}_0 = e'^{\widehat{0}}_i - e'^{\widehat{i}}_0, \quad g^{0i} = e^i_{\widehat{0}} - e^0_{\widehat{i}} = e'^i_{\widehat{0}} - e'^0_{\widehat{i}}, \\ T^{\widehat{0}}_{\widehat{i}} = e^{\widehat{0}}_i + e'^0_{\widehat{i}}, \quad T^{\widehat{i}}_{\widehat{0}} = e^{\widehat{i}}_0 + e'^i_{\widehat{0}}. \end{gathered} \tag{47}$$

Obviously, $\beta^i = T^{\widehat{0}}_{\widehat{i}} = T^{\widehat{i}}_{\widehat{0}}$ and the explicit expression for the relative velocity is

$$V^i = c\beta^i = c(e^{\widehat{0}}_i + e'^0_{\widehat{i}}) = c(e^{\widehat{0}}_i - e'^{\widehat{0}}_i). \tag{48}$$

To choose an orthonormal frame (tetrad) in physical terms means to select a local comoving reference system of an observer. Accordingly, one should be careful at this point and provide convincing physical arguments in favor of one's choice.

The tetrad is not definitely defined by a choice of the gauge. $\boldsymbol{V}$ can be equal to zero for two different tetrads satisfying the same gauge.

We use the Schwinger gauge [22] introduced also by Dirac [23] and characterized by the conditions $e_i^{\hat{0}} = 0$, $e_{\hat{i}}^0 = 0$. In this case, $e_{\hat{i}0} = e_{\hat{0}}^i = g_{0i}$. This gauge is the most convenient for a description of physical effects in gravitational fields and noninertial frames [10, 12, 13]. For any other gauge, the PK equation relates to a special reference frame. Such a frame moves relative to the laboratory frame with the velocity $\boldsymbol{V}$ defined by Eq. (48). Evidently, the relative motion of the two frames can be accelerated. In this case, the Thomas precession leads to the difference between the angular velocities of the rotation of the spin in the two frames [10].

For an investigation of gravitational phenomena, other gauges have also been proposed and used. The symmetric gauge has been applied by Pomeransky and Khriplovich [4] and in some other works [24, 25]. This gauge uses the tetrad which is invariant under the transposition of tetrad and world indices (former ones are marked by hats): $e_{\hat{\mu}\nu} = e_{\hat{\nu}\mu}$. This gauge is defined by $e_{\hat{0}i} = e_{\hat{i}0} = g_{0i}/2$.

The Landau-Lifshitz gauge [26] fixes the tetrads so that $e_0^{\hat{i}} = 0$, $e_{\hat{0}}^i = 0$, $e_{\hat{0}i} = g_{0i}$.

Either of the two last gauges distorts the description of spin motion. The situation can be explained by the example of a rotating frame [10]. The metric [14] is defined by Eq. (6) with $\boldsymbol{a} = 0$.

Only the use of the Schwinger gauge leads to the well-known Gorbatsevich-Mashhoon formula [27]:

$$\boldsymbol{\Omega} = -\boldsymbol{\omega}, \tag{49}$$

while the symmetric and Landau-Lifshitz gauges result in the equations

$$\boldsymbol{\Omega} = -\boldsymbol{\omega} + \frac{\widehat{\boldsymbol{u}} \times (\widehat{\boldsymbol{u}} \times \boldsymbol{\omega})}{2u^0(u^{\hat{0}} + 1)} \tag{50}$$

and

$$\boldsymbol{\Omega} = -\boldsymbol{\omega} + \frac{\widehat{\boldsymbol{u}} \times (\widehat{\boldsymbol{u}} \times \boldsymbol{\omega})}{u^0(u^{\hat{0}} + 1)}, \tag{51}$$

respectively [10]. Equation (49) is exact and equations (50) and (51) are obtained in the weak-field approximation. Evidently, the two last equations disagree with the Gorbatsevich-Mashhoon formula.

Similar situation takes place for equations of particle motion. For the Schwinger gauge, these equations take the form [10]

$$\frac{d\widehat{\boldsymbol{u}}}{dt} = -\boldsymbol{\omega} \times \widehat{\boldsymbol{u}}, \quad \frac{du^{\widehat{0}}}{dt} = 0. \tag{52}$$

Writing Eq. (52) in terms of the contravariant acceleration $du^{\mu}/(dt)$ leads to the right equation for the sum of the Coriolis and centrifugal accelerations (cf. Ref. [28]):

$$\frac{d\boldsymbol{u}}{dt} = -2\boldsymbol{\omega} \times \boldsymbol{u} - \frac{1}{c}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})u^0, \tag{53}$$

where $\boldsymbol{u} = \widehat{\boldsymbol{u}} - (\boldsymbol{\omega} \times \boldsymbol{r})u^{\widehat{0}}/c$ is the spatial part of the four-velocity u^{μ}. Other gauges bring misleading results. In the weak-field approximation, the symmetric gauge leads to the equation

$$\frac{d\widehat{\boldsymbol{u}}}{dt} = -\frac{3}{2}\boldsymbol{\omega} \times \widehat{\boldsymbol{u}}. \tag{54}$$

The Landau-Lifshitz gauge results in

$$\frac{d\widehat{\boldsymbol{u}}}{dt} = -2\boldsymbol{\omega} \times \widehat{\boldsymbol{u}}. \tag{55}$$

For any gauge, $u^{\widehat{0}}$ is constant.

An introduction of the gravitoelectromagnetic fields perfectly explains the origin of dependence of the equations of motion on the gauge. For the rotating frame, the Schwinger gauge brings the following exact formulas:

$$\boldsymbol{\mathcal{E}} = 0, \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\omega}u^{\widehat{0}}. \tag{56}$$

For other gauges, the gravitomagnetic field remains the same but the gravitoelectric field obtained in the weak-field approximation significantly differs:

$$\boldsymbol{\mathcal{E}}^{(sym)} = -\frac{1}{2}\boldsymbol{\omega} \times \widehat{\boldsymbol{u}}, \quad \boldsymbol{\mathcal{E}}^{(LL)} = -\boldsymbol{\omega} \times \widehat{\boldsymbol{u}}. \tag{57}$$

Equations (49),(50),(51),(52),(54), and (55) can be easily obtained from (56),(57) and general equations of motion (37),(38).

Similar conclusion results from the analysis of the spin motion in the Lense-Thirring [15] metric. The angular velocity of spin precession in the gravitational field of rotating source is given by [13]

$$\boldsymbol{\Omega} = \frac{G}{c^2 r^3}\left[\frac{3\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{J})}{r^2} - \boldsymbol{J}\right] + \frac{\boldsymbol{\rho}\times\boldsymbol{v}}{c^2}, \quad \boldsymbol{\rho} = \frac{2\gamma+1}{\gamma+1}\frac{GM}{r^3}\boldsymbol{r}$$
$$+\frac{\gamma}{\gamma+1}\frac{3G}{c^2 r^3}\left[\frac{\boldsymbol{r}}{r^2}(\boldsymbol{r}\cdot(\boldsymbol{J}\times\boldsymbol{v})) - \frac{2\xi}{3}\boldsymbol{J}\times\boldsymbol{v} + (2\xi-1)\frac{(\boldsymbol{r}\cdot\boldsymbol{v})}{r^2}\boldsymbol{J}\times\boldsymbol{r}\right], \tag{58}$$

where $\boldsymbol{v} = c\widehat{\boldsymbol{u}}/u^{\widehat{0}}$ is the particle velocity. This equation describes the geodetic precession and the LT one for the *relativistic* particle. Putting $\xi = 0$, we specify to the Schwinger tetrad, while $\xi = 1/2$ and $\xi = 1$ correspond to the symmetric tetrad (used in Refs. [4, 24, 25]) and the Landau-Lifshitz one [26], respectively.

Thus, the gravitoelectromagnetic fields and the equations of motion significantly depend on the gauge. As a result, the use of the Schwinger gauge is necessary and other gauges may bring misleading results.

6 Gravitoelectromagnetic fields for specific spacetimes

6.1 General noninertial frame

The accelerated and rotating noninertial frame presents the general case of inertial fields in a flat spacetime. An acceleration and an angular velocity of rotation of an observer are independent of the spatial coordinates but may depend arbitrarily on time. The metric of this frame defined by Eqs. (5) and (6) is given by [14]

$$ds^2 = \left[\left(1+\frac{\boldsymbol{a}(t)\cdot\boldsymbol{r}}{c^2}\right)^2 - \frac{[\boldsymbol{\omega}(t)\times\boldsymbol{r}]^2}{c^2}\right]c^2dt^2 - 2\boldsymbol{\omega}(t)\times\boldsymbol{r}\,dt - \delta_{ij}dx^i dx^j. \tag{59}$$

The nonzero Ricci rotation coefficients has the same form as for the

time-independent inertial fields:

$$\Gamma_{ij0} = -\frac{ce_{ijk}\omega^k(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}, \quad \Gamma_{0i0} = -\frac{a^i(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}. \tag{60}$$

The gravitoelectromagnetic fields are given by

$$\boldsymbol{\mathcal{E}} = -\frac{c\boldsymbol{a}(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}u^{\hat{0}}, \quad \boldsymbol{\mathcal{B}} = \frac{c^2\boldsymbol{\omega}(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}u^{\hat{0}}. \tag{61}$$

Amazingly, there is only the gravitoelectric field in the uniformly accelerated frame:

$$\boldsymbol{\mathcal{E}} = -\frac{c\boldsymbol{a}(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}u^{\hat{0}}, \quad \boldsymbol{\mathcal{B}} = 0 \tag{62}$$

and only the gravitomagnetic field in the rotating frame:

$$\boldsymbol{\mathcal{E}} = 0, \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\omega}(t)u^{\hat{0}}. \tag{63}$$

All equations presented in this subsection are *exact*.

Evidently, Eqs. (61),(62), and (63) agree with the nonrelativistic and weak-field approximation ensured by the conventional approach [Eq. (7)].

6.2 Lense-Thirring metric

Lense and Thirring have discovered in 1918 that rotating bodies "drag" the spacetime around themselves (frame dragging [15]). In other words, they have demonstrated the similarity between rotating frames and spacetimes created by rotating bodies. For the LT metric (8), the gravitoelectromagnetic fields read

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= -\frac{Gm}{cr^3}\boldsymbol{r}u^{\hat{0}} + \frac{3G}{c^2r^5}\left[\boldsymbol{r}(\widehat{\boldsymbol{u}}\cdot(\boldsymbol{J}\times\boldsymbol{r})) + (\boldsymbol{J}\times\boldsymbol{r})(\widehat{\boldsymbol{u}}\cdot\boldsymbol{r})\right],\\ \boldsymbol{\mathcal{B}} &= -\frac{Gm}{cr^3}\boldsymbol{r}\times\widehat{\boldsymbol{u}} - \frac{G}{c^2r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})\boldsymbol{r}}{r^2} - \boldsymbol{J}\right]u^{\hat{0}}.\end{aligned} \tag{64}$$

Because (see Ref. [13])

$$\begin{aligned}(\widehat{\boldsymbol{u}}\times\boldsymbol{r})(\widehat{\boldsymbol{u}}\cdot(\boldsymbol{J}\times\boldsymbol{r})) + (\widehat{\boldsymbol{u}}\times(\boldsymbol{J}\times\boldsymbol{r}))(\widehat{\boldsymbol{u}}\cdot\boldsymbol{r}) &= -2(\boldsymbol{r}\times\widehat{\boldsymbol{u}})(\boldsymbol{J}\cdot(\boldsymbol{r}\times\widehat{\boldsymbol{u}}))\\ -(\widehat{\boldsymbol{u}}\times(\boldsymbol{r}\times\widehat{\boldsymbol{u}}))(\boldsymbol{r}\cdot\boldsymbol{J}) - r^2(\widehat{\boldsymbol{u}}\times(\widehat{\boldsymbol{u}}\times\boldsymbol{J})),&\end{aligned}$$

Eqs. (37) and (64) lead to Eq. (58) (for $\xi = 0$).

Equation (64) is in full agreement with the approximate formula (11). However, one of important preferences of the relativistic approach is a discovery of dependence of $\boldsymbol{\mathcal{E}}$ on nondiagonal and $\boldsymbol{\mathcal{B}}$ on diagonal components of the metric tensor. This property does not follow from the conventional conception of gravitoelectromagnetism (see Sec. 2).

6.3 Static gravitational fields in isotropic coordinates

The static metric in the isotropic coordinates is defined by Eq. (5) at condition that $\boldsymbol{K} = 0$ and has the form

$$ds^2 = V^2 c^2 dt^2 - W^2 (d\boldsymbol{r} \cdot d\boldsymbol{r}). \tag{65}$$

The respective gravitoelectromagnetic fields are given by

$$\boldsymbol{\mathcal{E}} = -cu^{\hat{0}} \nabla V, \quad \boldsymbol{\mathcal{B}} = c \nabla W \times \widehat{\boldsymbol{u}}. \tag{66}$$

An appearance of nonzero $\boldsymbol{\mathcal{B}}$ depending on W is a new property as compared with the conventional conception of gravitoelectromagnetism.

The most important examples of static fields are the Schwarzschild, de Sitter and anti-de Sitter spacetimes. For the Schwarzschild metric in the isotropic coordinates in the weak-field approximation, V and W are given by Eq. (8) and the gravitoelectromagnetic fields become

$$\boldsymbol{\mathcal{E}} = -\frac{GM\boldsymbol{r}}{cr^3} u^{\hat{0}} = \frac{\boldsymbol{g} u^{\hat{0}}}{c}, \quad \boldsymbol{\mathcal{B}} = -\frac{GM}{cr^3} \boldsymbol{r} \times \widehat{\boldsymbol{u}} = \frac{\boldsymbol{g} \times \widehat{\boldsymbol{u}}}{c}, \tag{67}$$

where $\boldsymbol{g}$ is the Newtonian acceleration.

The four-dimensional de Sitter metric can be presented in the form

$$ds^2 = \left(1 - \frac{r^2}{\alpha^2}\right) c^2 dt^2 - \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{68}$$

There is a cosmological horizon at $r = \alpha$.

De Sitter space is an Einstein manifold since the Ricci tensor is proportional to the metric:

$$R_{\mu\nu} = \frac{4}{\alpha^2} g_{\mu\nu}.$$

This means the de Sitter spacetime is a vacuum solution of the Einstein equation with the cosmological constant $\Lambda = 3/\alpha^2$ and the scalar curvature $R = 4\Lambda = 12/\alpha^2$.

The anti-de Sitter metric can be obtained with the substitution $\alpha \to ik$.

The well-known coordinate transformation may reduce de Sitter and anti-de Sitter metrics to isotropic forms. For the de Sitter metric, this transformation is given by

$$r = \frac{\rho}{1 + \frac{\rho^2}{4\alpha^2}}. \tag{69}$$

The metric takes the form

$$ds^2 = k_-^2 k_+^{-2} c^2 dt^2 - k_+^{-2} \left[d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2) \right], \quad k_\pm = 1 \pm \frac{\rho^2}{4\alpha^2}. \tag{70}$$

As a result, the isotropic Cartesian coordinates can be used.

The gravitoelectromagnetic fields read

$$\boldsymbol{\mathcal{E}} = \frac{c}{\alpha^2} \boldsymbol{\rho} u^{\widehat{0}}, \quad \boldsymbol{\mathcal{B}} = -\frac{c}{\alpha^2} \boldsymbol{\rho} \times \widehat{\boldsymbol{u}}. \tag{71}$$

For the anti-de Sitter metric, the gravitoelectromagnetic fields can be obtained with the substitution $\alpha \to ik$ and are equal to

$$\boldsymbol{\mathcal{E}} = -\frac{c}{k^2} \boldsymbol{\rho} u^{\widehat{0}}, \quad \boldsymbol{\mathcal{B}} = \frac{c}{k^2} \boldsymbol{\rho} \times \widehat{\boldsymbol{u}}. \tag{72}$$

An existence of the gravitomagnetic field is very important for an evolution of momentum and spin in static gravitational fields.

7 Mathisson force

The great achievement of Myron Mathisson was the discovery of an additional force acting on a spinning particle in a curved spacetime. The Mathisson force is similar to the Stern-Gerlach one in electrodynamics but specifically depends on a particle motion. An analysis of this force can be fulfilled in the general case (see, e.g., Refs. [9, 21]). Nevertheless, a

derivation of a simple expression in the weak-field approximation seems to be rather important. It states a deep similarity between the gravity and electromagnetism existing even for effects depending on curvature.

To take into account an influence of the spin on the particle motion, we may use the Hamiltonian method and add the Hamiltonian of a spinless particle in a gravitational field [29] by the term $\boldsymbol{s} \cdot \boldsymbol{\Omega}$:

$$\mathcal{H} = \mathcal{H}_0 + \boldsymbol{s} \cdot \boldsymbol{\Omega}. \tag{73}$$

The possibility of this addition has been mentioned in Ref. [4] and has been rigorously substantiated not only in framework of classical gravity but also for Dirac particles in Refs. [12, 21].

It follows from Eqs. (37) and (73) that the additional force acting on the spinning particle is given by

$$\boldsymbol{f}_M = -\nabla(\mathcal{H} - \mathcal{H}_0) = -\nabla\left(\frac{1}{u^0}\boldsymbol{s} \cdot \left[-\boldsymbol{\mathcal{B}} + \frac{\widehat{\boldsymbol{u}} \times \boldsymbol{\mathcal{E}}}{u^{\widehat{0}} + 1}\right]\right). \tag{74}$$

In electrodynamics, the Stern-Gerlach force *acting on a Dirac particle* has the same form:

$$\boldsymbol{f}_{SG} = -\frac{e}{mc}\nabla\left(\boldsymbol{s} \cdot \left[-\boldsymbol{B} + \frac{\boldsymbol{u} \times \boldsymbol{E}}{u^0 + 1}\right]\right). \tag{75}$$

Equation (75) is obtained in the classical limit. The most important difference between Eqs. (74) and (75) consists in the dependence of the gravitoelectromagnetic fields on the four-velocity.

Since the Stern-Gerlach and Mathisson forces are proportional to gradients of scalars, they bring the additions to the electric and gravitoelectric forces, respectively. This property has been found in Ref. [21]. As a result, $\nabla \times \boldsymbol{f}_{SG} = 0$, $\nabla \times \boldsymbol{f}_M = 0$.

The Mathisson force violates the weak equivalence principle [9] because particles with different spin orientations move on different trajectories. Since the gravitoelectromagnetic fields are proportional to derivatives of the metric, the Mathisson force is proportional to the curvature (see Ref. [21]).

The resulting force acting on the spinning particle is given by

$$\boldsymbol{F} = \frac{mcu^{\widehat{0}}}{u^0}\left(\boldsymbol{\mathcal{E}} + \frac{\widehat{\boldsymbol{u}} \times \boldsymbol{\mathcal{B}}}{u^{\widehat{0}}}\right) + \boldsymbol{f}_M. \tag{76}$$

Evidently, this equation presents the next order approximation as compared with the PK equation (38).

8 Discussion and summary

The introduction of the gravitoelectromagnetic fields being the fields in the local Lorentz frame significantly simplifies a description of motion of spinning particles in the general relativity. When the mutual influence of particle and spin motion is neglected, dynamics of the four-velocity and spin is defined by Eqs. (37) and (38) similar to corresponding equations in electrodynamics. The transition to the world coordinates is not difficult because

$$u^\mu = e^\mu_a u^a, \quad \frac{du^\mu}{dt} = e^\mu_a \frac{du^a}{dt} + \left(\frac{\partial e^\mu_a}{\partial t} + v^i e^\mu_{a,\,i} \right) u^a.$$

Moreover, the investigation of the particle motion is simplified when the trajectory is infinite. In this case, one can apply the fact that the four-vectors u^μ and u^a coincide at the initial and final parts of such a trajectory of the particle because of the very large distance to the field source [12].

The comparison with the conventional conception of gravitoelectromagnetism demonstrates great possibilities of the new relativistic conception in description of dynamics of spinning particles in the general relativity. This conception uses equations of motion in an appropriate anholonomic frame and is based on the Pomeransky-Khriplovich gravitoelectromagnetic fields and the Schwinger gauge. It is applicable for relativistic spinning particles in arbitrarily strong gravitational fields and perfectly describes the evolution of momentum and spin. The dynamics of the three-component spin is defined in an anholonomic frame. For consistency, we also apply the anholonomic frame for determination of the particle dynamics. The weak-field approximation used in the present work is sufficient to show preferences of the new conception in description of relativistic effects. In addition, this approximation allows us to obtain very simple explicit expressions for the gravitoelectromagnetic fields.[1)]

[1)] The explicit expressions for the gravitoelectromagnetic fields in strong gravitational fields have been derived in Refs. [12, 21].

The considered example of the rotating frame clarifies an importance of the Schwinger gauge and an inconvenience of other gauges (including the symmetric gauge used in earlier investigations [4, 24, 25]). We have calculated the gravitoelectromagnetic fields for the mainly used spacetimes. Finally, we have used these fields in order to derive the simple expression for the Mathisson force in the weak-field approximation.

The introduction of the gravitoelectromagnetic fields applicable for exact description of dynamics of relativistic spinning particles in arbitrary spacetimes can be regarded as a new powerful mathematical method in general relativity. Comparatively simple equations for the four-velocity and spin and clear analogy between electromagnetism and gravity are the main distinctive features of this general method.

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FUZZY 3-LIE ALGEBRA

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Abstract

In this paper, we have tried to apply the concepts of fuzzy sets to 3-Lie algebra, then we define some concepts related to fuzzy sets and discuss their properties. In the following paper we introduce and study a new sort of fuzzy 3-Lie algebra of a 3-Lie algebra called $(\in, \in \vee q)$-fuzzy 3-Lie algebra. These fuzzy 3-Lie algebras are characterized by their level subalgebras. Finally, we give a generalization of $(\in, \in \vee q)$-fuzzy 3-Lie algebra.

Key words: 3-Lie algebras; Fuzzy sets; Fuzzy 3-Lie algebras; Fuzzy 3-Lie ideals; level set; $(\in, \in \vee q)$-fuzzy 3-Lie algebra.

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§1 Introduction

The theory of fuzzy sets which was introduced by Zadeh is applied to many mathematical branches. Since Zadeh formulated the notion of a fuzzy set, fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. Rosenfeld inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups. Das characterized fuzzy subgroups by their level subgroups. A new type of fuzzy subgroup ($(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets.

Algebraic structures play an important role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting.

In this paper we give the definition of fuzzy 3-Lie algebra. Then we give some properties of fuzzy 3-Lie algebra by the level sets. At last we introduce and study a new sort of fuzzy 3-Lie algebra of a 3-Lie algebra called $(\in, \in \vee q)$-fuzzy 3-Lie algebra.

Let X be a set. A fuzzy set μ of X is a function $\mu : X \to [0, 1]$. If μ and λ are two fuzzy sets of X, then we say $\mu \subseteq \lambda$ if $\mu(x) \leq \lambda(x)$ for every $x \in X$. The set $\{x \in X : \mu(x) \geq t\}$ is called a level subset of μ and is symbolized by $[\mu]_t$.

Let $\mu_i : i \in I$ be fuzzy sets of X. Union and intersection of fuzzy sets $\mu_i : i \in I$ are defined by $(\bigcup_{i\in I} \mu_i)(x) = \bigvee_{i\in I} \mu_i(x)$ and $(\bigcap_{i\in I} \mu_i)(x) = \bigwedge_{i\in I} \mu_i(x)$.

§2 Fuzzy 3-Lie algebras and fuzzy 3-Lie ideals

In this section, we extend the concept of 3-Lie algebra to the framework of fuzzy setting and define a fuzzy 3-Lie algebra and a fuzzy 3-Lie ideal and discuss their properties.

Definition 2.1. A vector space L over a field F is an 3-Lie algebra if there is an 3-ary multilinear operation $[,,]$ satisfying the following identities:

$$\begin{aligned} (1) \qquad [x_1, x_2, x_3] &= (-1)^{\tau(\sigma)}[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}], \\ (2)\ [[x_1, x_2, x_3], y_2, y_3] &= [[x_1, y_2, y_3], x_2, x_3] + [x_1, [x_2, y_2, y_3], x_3] \\ &+ [x_1, x_2, [x_3, y_2, y_3]], \end{aligned}$$

where σ runs over the symmetric group S_n and the number $\tau(\sigma)$ is equal to 0 or 1 depending on the parity of the permutaion σ.

From now on, L is a 3-Lie algebra over a field F. A subspace K of L is called a 3- Lie subalgebra of L if $[x, y, z] \in K$, for all $x, y, z \in K$ and a subspace I of L is called a 3-Lie ideal of L if for all $x \in I, y, z \in L$ implies $[x, y, z] \in I$.

Definition 2.2. A fuzzy set μ of L is called a fuzzy 3-Lie algebra of L, if for all $x, y, z \in L, \alpha \in F$, the following requirements are met:

(1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$

(2) $\mu(\alpha x) \geq \mu(x)$

(3) $\mu([x, y, z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$

If the condition (3) is replaced by $\mu([x, y, z]) \geq \mu(x) \vee \mu(y) \vee \mu(z)$, then μ is called a fuzzy 3-Lie ideal of L.

Proposition 2.3. Let μ be a fuzzy 3-Lie algebra of L,then

(1) $\mu(\alpha x) = \mu(x)$, for all $0 \neq \alpha \in F$, for all $x \in L$, in particular, $\mu(-x) = \mu(x)$

(2) $\mu(0) \geq \mu(x)$, for all $x \in L$

(3) $\mu(x) < \mu(y) \Rightarrow \mu(x-y) = \mu(x) = \mu(y-x)$

(4) $\mu([x,y,z]) = \mu(-[y,x,z]) = \mu([y,x,z])$

Proof. (1) As μ is a fuzzy 3-Lie algebra of L. Then we have $\mu(\alpha x) \geq \mu(x)$. If $\alpha \neq 0$, then $\mu(x) = \mu(\frac{1}{\alpha} \cdot \alpha x) \geq \mu(\alpha x)$. So $\mu(x) = \mu(\alpha x)$. In particular,$\mu(-x) = \mu(x)$ when α is -1.

(2) We know $\mu(0) = \mu(x-x) \geq \mu(x) \wedge \mu(-x) = \mu(x) \wedge \mu(x) = \mu(x)$ for every $x \in L$.

(3) By (2) we have $\mu(x-y) = \mu(y-x)$, so we only need to prove $\mu(x-y) = \mu(x)$. If $\mu(x) < \mu(y)$, then $\mu(x-y) \geq \mu(x) \wedge \mu(-y) = \mu(x) \wedge \mu(y) = \mu(x)$. We also have $\mu(x) = \mu(x-y+y) \geq \mu(x-y) \wedge \mu(y)$, then $\mu(x) \geq \mu(x-y)$ by $\mu(x) \geq \mu(y)$. So we have $\mu(x) = \mu(x-y)$

(4) By (3) this conclusion is trivial.

Proposition 2.4. Let L be a 3-Lie algebra and χ_I be the characteristic function of a subset I of L. Then χ_I is a fuzzy 3-Lie algebra(ideal) if and only if I is a subalgebra(ideal) of L,respectively.

Proof. First we prove the condition of subalgebra.

($\Leftarrow$) First we know χ_I is a fuzzy set of L. Because I is a subalgebra of L. Then $x+y \in I$,$\alpha x \in I$ and $[x,y,z] \in I$ for all $x,y,z \in I, \alpha \in F$. So we have $\mu(x+y) = 1 \geq 1 = \mu(x) \wedge \mu(y)$, $\mu([x,y,z]) = 1 \geq 1 = \mu(x) \wedge \mu(y) \wedge \mu(z)$. And if $x \notin I$ or $y \notin I$, the conclusion is trivial. So μ is a fuzzy 3-Lie algebra of L.

($\Rightarrow$) Suppose χ_I is a fuzzy 3-Lie algebra. Then for all $x,y,z \in I, \alpha \in F$. If $x+y \notin I$. Then $0 = \mu(x+y) \geq \mu(x) \wedge \mu(y) = 1$, a contradiction. If $\alpha x \notin I$. Then $0 = \mu(\alpha x) \geq \mu(x) = 1$, a contradiction. If $[x,y,z] \notin I$. Then $0 = \mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) = 1$, a contradiction. So I is a 3-Lie algebra.

Next we prove the condition of ideal.

($\Leftarrow$) Let I be an ideal of L. Then for all $x \in I, y,z \in L, \alpha \in F$, we have

$[x, y, z] \in I$. So $\mu([x, y, z]) = 1 \geq 1 = \mu(x) \vee \mu(y) \vee \mu(z)$. And if $x, y, z \notin I$, the conclusion is trivial. So χ_I is a fuzzy ideal.

($\Rightarrow$) Suppose χ_I is a fuzzy ideal . Then for all $x, y \in L, z \in I, \alpha \in F$. If $[x, y, z] \notin I$. Then $0 = \mu([x, y, z]) \geq \mu(x) \vee \mu(y) \vee \mu(z) = 1$, which is a contradiction. So χ_I is a fuzzy ideal.

Theorem 2.5. A fuzzy set μ of L is a fuzzy 3-Lie algebra(ideals) of L if and only if the level subsets $[\mu]_t = \{x \in L : \mu(x) \geq t\}$, for $0 \leq t \leq \mu(0)$, are 3-Lie algebras(ideals) of L ,respectively.

Proof. ($\Rightarrow$) Let $0 \leq t \leq \mu(0)$. Since $0 \in [\mu]_t$, we have $[\mu]_t \neq \emptyset$. Let $x, y, z \in [\mu]_t$ and $\alpha \in F$. Since $\mu(x + y) \geq \mu(x) \wedge \mu(y) \geq t$, we have $x + y \in [\mu]_t$. Since $\mu(\alpha x) \geq \mu(x) \geq t$, we have $\alpha x \in [\mu]_t$. Since $\mu([x, y, z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \geq t$, we have $[x, y, z] \in [\mu]_t$. Thus $[\mu]_t$ is a 3-Lie subalgebra of L.

($\Leftarrow$) Let $x, y, z \in L$ and $\alpha \in F$. We may assume that $\mu(z) \geq \mu(y) \geq \mu(x) = t$, so we have $x, y, z \in [\mu]_t$. Since $[\mu]_t$ is a 3-Lie subalgebra of L, we have $x + y, \alpha x, [x, y, z] \in [\mu]_t$. Thus we have $\mu(x + y) \geq t = \mu(x) \wedge \mu(y)$, $\mu(\alpha x) \geq t = \mu(x)$ and $\mu([x, y, z]) \geq t = \mu(x) \wedge \mu(y) \wedge \mu(z)$. Hence μ is a fuzzy 3-Lie algebra of L.

Now consider the cases of fuzzy 3-Lie ideals of L.

($\Rightarrow$) Let $x, y \in L, z \in [\mu]_t$. Since $\mu([x, y, z]) \geq \mu(x) \vee \mu(y) \vee \mu(z) \geq \mu(z) \geq t$, we have $[x, y, z] \in [\mu]_t$. Hence $[\mu]_t$ is a 3-Lie ideal of L.

($\Leftarrow$) Let $x, y, z \in L$, We may assume that $\mu(x) \leq \mu(y) \leq \mu(z) = t$, so we have $z \in [\mu]_t$. Since $[\mu]_t$ is a 3-Lie ideal of L, we have $[x, y, z] \in [\mu]_t$. Thus we have $\mu([x, y, z]) \geq t = \mu(x) \vee \mu(y) \vee \mu(z)$. Hence μ is a fuzzy 3-Lie ideal of L.

We now redefine the sup-min product of two fuzzy 3-Lie ideals of L. The redefined product is a modification of the sup-min product.

Definition 2.6. Let ν_1, ν_2, ν_3 be any three fuzzy sets of L. The sup-min product $[[\nu_1\nu_2\nu_3]]$ of ν_1 , ν_2 and ν_3 is defined as follows, for all $x \in L$,

$$[[\nu_1\nu_2\nu_3]](x) = \begin{cases} \vee\{\bigwedge_{i=1}^{m}\{\nu_1(c_i) \wedge \nu_2(d_i) \wedge \nu_3(e_i)\} \mid x = \sum_{i=1}^{m}[c_i, d_i, e_i]\}, \\ \text{where } c_i, d_i, e_i \in L \\ 0, \text{ if x is not expressible as } x = \sum_{i=1}^{m}[c_i, d_i, e_i]. \end{cases}$$

Proposition 2.7. Let L be a 3-Lie algebra which dimension is finite. ν_1, ν_2, ν_3 are three fuzzy 3-Lie algebra of L, where $t \in [0, 1]$. Then we have: $[[[\nu_1\nu_2\nu_3]]]_t = [[\nu_1]_t, [\nu_2]_t, [\nu_3]_t]$, for all $t \in [0, 1]$.

Proof. As L is a 3-Lie algebra of finite dimension. Then fuzzy sets ν_1, ν_2, ν_3 take finite value by $\mu(\alpha x) = \mu(x)$. Then we can take three groups real numbers as follows: $\{r_0 = 0 < r_1 < \cdots < r_{m-1} < 1 = r_m\}$, $\{s_0 = 0 < s_1 < \cdots < s_{m-1} < 1 = s_n\}$, and $\{l_0 = 0 < l_1 < \cdots < l_{m-1} < 1 = l_p\}$, satisfies for all $r \in [r_{i-1}, r_i], s \in [s_{j-1}, r_j], q \in [l_{k-1}, l_k]$, we have $[\nu_1]_i = [\nu_1]_{r_i}(i = 1, 2, \cdots, m)$, $[\nu_2]_s = [\nu_2]_{s_j}(j = 1, 2, \cdots, n)$ and $[\nu_3]_q = [\nu_3]_{l_k}(k = 1, 2, \cdots, p)$.

Then for all $t \in [0, 1]$, $x \in [[[\nu_1\nu_2\nu_3]]]_t$. We may assume $[[[\nu_1\nu_2\nu_3]]](x) = \alpha \geq t$, and we have $1 \leq i \leq m$,$1 \leq j \leq n$ and $1 \leq k \leq p$ satisfies $r_{i-1} < \alpha \leq r_i$, $s_{j-1} < \alpha \leq s_j$ and $l_{k-1} < \alpha \leq l_k$. So $(r_{i-1} \vee s_{j-1} \vee l_{k-1}) < \alpha \leq r_i \vee s_j \vee l_k$. Then there exists $c, d, e \in L$ satisfies $x = [c, d, e]$, and $\nu_1(c) \wedge \nu_2(d) \wedge \nu_3(e) > r_{i-1} \vee s_{j-1} \vee l_{k-1}$. Then $\nu_1(c) \geq r_i$, $\nu_2(d) \geq s_j$ and $\nu_3(e) \geq l_k$, hence $\nu_1(c) \wedge \nu_2(d) \wedge \nu_3(e) \geq r_i \wedge s_j \wedge l_k \geq \alpha$. Because of $\alpha = [[\nu_1\nu_2\nu_3]](x) = \nu_1(c) \wedge \nu_2(d) \wedge \nu_3(e) \geq \alpha$. Then $[[\nu_1\nu_2\nu_3]](x) = \nu_1(c) \wedge \nu_2(d) \wedge \nu_3(e)$, where $c \in [\nu_1]_t, d \in [\nu_2]_t, e \in [\nu_3]_t$. So $[[[\nu_1\nu_2\nu_3]]]_t \subseteq [[\nu_1]_t, [\nu_2]_t, [\nu_3]_t]$. As $[[[\nu_1\nu_2\nu_3]]]_t \supseteq [[\nu_1]_t, [\nu_2]_t, [\nu_3]_t]$ is clear,then we have $[[[\nu_1\nu_2\nu_3]]]_t = [[\nu_1]_t, [\nu_2]_t, [\nu_3]_t]$.

Theorem 2.8. Let ν_1, ν_2, ν_3 be any three fuzzy 3-Lie ideals of L. Then the sup-min product $[[\nu_1\nu_2\nu_3]]$ is also fuzzy 3-Lie ideal of L.

Proof. Suppose $x, y, z \in L$. We have to show that Definition 2.2 is

satisfied.

(1) We prove that $[[\nu_1\nu_2\nu_3]](x+y) \geq [[\nu_1\nu_2\nu_3]](x) \wedge [[\nu_1\nu_2\nu_3]](y)$.

Suppose that $[[\nu_1\nu_2\nu_3]](x+y) < [[\nu_1\nu_2\nu_3]](x) \wedge [[\nu_1\nu_2\nu_3]](y)$. Then we have $[[\nu_1\nu_2\nu_3]](x+y) < [[\nu_1\nu_2\nu_3]](x)$ and $[[\nu_1\nu_2\nu_3]](x+y) < [[\nu_1\nu_2\nu_3]](y)$. Choose a number $t \in [0,1]$ such that $[[\nu_1\nu_2\nu_3]](x+y) < t < [[\nu_1\nu_2\nu_3]](x)$ and $[[\nu_1\nu_2\nu_3]](x+y) < t < [[\nu_1\nu_2\nu_3]](y)$.

There exist $x_i, y_i, z_i, x_j^1, y_j^1, z_j^1 \in L$ such that $x = \sum_{i=1}^{n}[x_i, y_i, z_i]$ and $y = \sum_{j=1}^{m}[x_j^1, y_j^1, z_j^1]$ and for all i, j, $\nu_1(x_i) > t, \nu_2(y_i) > t, \nu_3(z_i) > t$, $\nu_1(x_i^1) > t, \nu_2(y_i^1) > t, \nu_3(z_i^1) > t$.

Since $x+y = \sum_{i=1}^{n}[x_i, y_i, z_i] + \sum_{j=1}^{m}[x_j^1, y_j^1, z_j^1]$, then we have $[[\nu_1\nu_2\nu_3]](x+y) > t$. But $[[\nu_1\nu_2\nu_3]](x+y) < t$, this is a contradiction.

(2) Let $\alpha \in F$ and $x \in L$. By the proof of (1) it is clear that $[[\nu_1\nu_2\nu_3]](\alpha x) \geq [[\nu_1\nu_2\nu_3]](x)$.

(3) We prove that $[[\nu_1\nu_2\nu_3]]([x,y,z]) \geq [[\nu_1\nu_2\nu_3]](x) \vee [[\nu_1\nu_2\nu_3]](y) \vee [[\nu_1\nu_2\nu_3]](z)$. Suppose that $[[\nu_1\nu_2\nu_3]]([x,y,z]) < [[\nu_1\nu_2\nu_3]](x) \vee [[\nu_1\nu_2\nu_3]](y) \vee [[\nu_1\nu_2\nu_3]](z)$.

Then we have $[[\nu_1\nu_2\nu_3]]([x,y,z]) < [[\nu_1\nu_2\nu_3]](x)$ or $[[\nu_1\nu_2\nu_3]]([x,y,z]) < [[\nu_1\nu_2\nu_3]](y)$ or $[[\nu_1\nu_2\nu_3]]([x,y,z]) < [[\nu_1\nu_2\nu_3]](z)$. Let $[[\nu_1\nu_2\nu_3]]([x,y,z]) < [[\nu_1\nu_2\nu_3]](x)$. Choose a number $t \in [0,1]$ such that $[[\nu_1\nu_2\nu_3]]([x,y,z]) < t < [[\nu_1\nu_2\nu_3]](x)$.

There exist $x_i, y_i, z_i \in L$ such that $x = \sum_{i=1}^{n}[x_i, y_i, z_i]$ and for all i, $\nu_1(x_i) > t, \nu_2(y_i) > t, \nu_3(z_i) > t$. Since $[x,y,z] = [\sum_{i=1}^{n}[x_i, y_i, z_i], y, z]$, then we have

$$[[\nu_1\nu_2\nu_3]]([x,y,z]) = [[\nu_1\nu_2\nu_3]]([\sum_{i=1}^{n}[x_i, y_i, z_i], y, z]) = [[\nu_1\nu_2\nu_3]](\sum_{i=1}^{n}[[x_i, y_i, z_i], y, z])$$
$$\geq [[\nu_1\nu_2\nu_3]]([[x_i, y_i, z_i], y, z])$$
$$= [[\nu_1\nu_2\nu_3]]([x_i, y_i, [z_i, y, z]] + [x_i, [y_i, y, z], z_i] + [[x_i, y, z], y_i, z_i])$$
$$\geq [[\nu_1\nu_2\nu_3]]([x_i, y_i, [z_i, y, z]])$$

$\geq \wedge\{\nu_1(x_i), \nu_2(y_i), \nu_3([z_i, y, z])\}$

$\geq \wedge\{\nu_1(x_i), \nu_2(y_i), \vee(\nu_3(z_i), \nu_3(y), \nu_3(z))\}$

$> t$

Then we have $[[\nu_1\nu_2\nu_3]]([x, y, z]) > t > [[\nu_1\nu_2\nu_3]]([x, y, z])$, this is a contradiction.

In the case of $[[\nu_1\nu_2\nu_3]]([x, y, z]) < [[\nu_1\nu_2\nu_3]](y)$ and $[[\nu_1\nu_2\nu_3]]([x, y, z]) < [[\nu_1\nu_2\nu_3]](z)$, we get a contradiction, using the similar method. Hence $[[\nu_1\nu_2\nu_3]]$ is a fuzzy 3-Lie ideal of L.

§3 $(\in, \in \vee q)$-fuzzy 3-Lie algebra

For any fuzzy subset μ of L, the $\{x \in L \mid \mu(x) > 0\}$ is called the support of μ, and is denoted by $supp\mu$. A fuzzy set μ, on L which takes the value $t \in (0, 1]$ at some $x \in L$ and takes the value 0 for all $y \in L$ expect x is called a fuzzy point and is denoted by x_t, where the point x is called its support point and t is called its value. A fuzzy point x_t is said to belong to or quasi-coincident with a fuzzy set μ, written as $x_t \in \mu$ or $x_t q \mu$ if $\mu(x) \geq t$ or $\mu(x) + t > 1$. If $x_t \in \mu$ or $x_t q \mu$, then we write $x_t \in \vee q\mu$. The symbol $\overline{\in \vee q\mu}$ means $\in \vee q\mu$ dose not hold.

Definition 3.1. A fuzzy subset μ of a 3-Lie algebra L is said to be an $(\in, \in \vee q)$-fuzzy 3- Lie algebra of L if for all $s, t, r \in (0, 1], x, y, z \in L$ and $\alpha \in F$.

(1) $x_t, y_r \in \mu \Rightarrow (x + y)_{t \wedge r} \in \vee q\mu$

(2) $x_t \in \mu \Rightarrow (\alpha x)_t \in \vee q\mu$

(3) $x_t, y_r, z_s \in \mu \Rightarrow ([x, y, z])_{t \wedge r \wedge s} \in \vee q\mu$

Theorem 3.2. Conditions (1)-(3) in Definition 3.1 are equivalent to the following conditions respectively.

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$ for all $x, y \in L$,

(ii) $\mu(\alpha x) \geq \mu(x) \wedge 0.5$ for all $x \in L, \alpha \in F$,

(iii) $\mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5$ for all $x, y, z \in L$.

Proof. (1) $\Rightarrow$ (i): Suppose that $x, y \in L$. We consider the following cases:

(a) $\mu(x) \wedge \mu(y) < 0.5$

(b) $\mu(x) \wedge \mu(y) \geq 0.5$

Case a: Assume that $\mu(x+y) < \mu(x) \wedge \mu(y) \wedge 0.5$, which implies $\mu(x+y) < \mu(x) \wedge \mu(y)$. Choose t such that $\mu(x+y) < t < \mu(x) \wedge \mu(y)$. So $\mu(x) > t, \mu(y) > t$ and then $x_t \in \mu, y_t \in \mu$. By (1) we have $(x+y)_{t \wedge t} \in \vee q\mu$. But $\mu(x+y) < t < 0.5$ and $\mu(x+y) + t \wedge t < 1$, so $(x+y)_t \overline{\in \vee q}\mu$, this is a contradiction.

Case b: Assume that $\mu(x+y) < 0.5$. We have $\mu(x) \wedge \mu(y) \geq 0.5$, so $x_{0.5}, y_{0.5} \in \mu$. By (1) we have $(x+y)_{0.5} \in \vee q\mu$. But $\mu(x+y) < 0.5$ and $\mu(x+y) + 0.5 < 1$, so $(x+y)_{0.5} \overline{\in \vee q}\mu$, this is a contradiction.

(2) $\Rightarrow$ (ii): Suppose that $x \in L$. We consider the following cases:

(a) $\mu(x) < 0.5$

(b) $\mu(x) \geq 0.5$

Case a: Assume that $\mu(x) = t < 0.5$ and $\mu(\alpha x) = r < \mu(x)$. Choose s such that $r < s < t$ and $r + s < 1$. Then $x_s \in \mu$ but $(\alpha x)_s \overline{\in \vee q}\mu$, which contradicts (2). So $\mu(\alpha x) \geq \mu(x) = \mu(x) \wedge 0.5$.

Case b: Let $\mu(x) \geq 0.5$. If $\mu(\alpha x) < \mu(x) \wedge 0.5$, then $x_{0.5} \in \mu$, but $(\alpha x)_{0.5} \overline{\in \vee q}\mu$ which contradicts (2). So $\mu(\alpha x) \geq \mu(x) \wedge 0.5$.

(3) $\Rightarrow$ (iii): Suppose that $x, y, z \in L$. We consider the following cases:

(a) $\mu(x) \wedge \mu(y) \wedge \mu(z) < 0.5$

(b) $\mu(x) \wedge \mu(y) \wedge \mu(z) \geq 0.5$

Case a: Assume that $\mu([x,y,z]) < \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5$, which implied $\mu([x,y,z]) < \mu(x) \wedge \mu(y) \wedge \mu(z)$. Choose t such that $\mu([x,y,z]) < t < \mu(x) \wedge \mu(y) \wedge \mu(z)$. So $\mu(x) > t$, $\mu(y) > t$ and $\mu(z) > t$. Then $x_t \in \mu$, $y_t \in \mu$ and $z_t \in \mu$, but $[x,y,z]_t \overline{\in \vee q}\mu$, which contradicts (3).

Case b: Assume that $\mu([x,y,z]) < 0.5$. Then $x_{0.5} \in \mu$, $y_{0.5} \in \mu$ and $z_{0.5} \in \mu$, but $[x,y,z]_{0.5}\overline{\in \vee q}\mu$, this is a contradiction.

$(i) \Rightarrow (1)$: Let $x_t, y_r \in \mu$, then $\mu(x) \geq t$ and $\mu(y) \geq r$. Now, we have $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge r \wedge 0.5$. If $t \wedge r > 0.5$, then $\mu(x+y) \geq 0.5$ which implies $\mu(x+y) + t \wedge r > 1$.

If $t \wedge r \leq 0.5$, then $\mu(x+y) \geq t \wedge r$. Therefore $(x+y)_{t \wedge r} \in \vee q\mu$.

$(ii) \Rightarrow (2)$: Let $x_t \in \mu$. Then $\mu(x) \geq t$. Now, we have $\mu(\alpha x) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5$, which implies $\mu(\alpha x) \geq t$ or $\mu(\alpha x) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5$. Therefore $(\alpha x)_t \in \vee q\mu$.

$(iii) \Rightarrow (3)$: Let $x_t, y_r, z_s \in \mu$, then $\mu(x) \geq t$, $\mu(y) \geq r$ and $\mu(z) \geq s$. Now we have $\mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5 \geq t \wedge r \wedge s \wedge 0.5$.

If $t \wedge r \wedge s > 0.5$, then $\mu([x,y,z]) \geq 0.5$, which implies $\mu([x,y,z]) + t \wedge r \wedge s > 1$.

If $t \wedge r \wedge s \leq 0.5$, then $\mu([x,y,z]) \geq t \wedge r \wedge s$. Therefore $([x,y,z])_{t \wedge r \wedge s} \in \vee q\mu$.

A fuzzy 3-Lie algebra is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra. Clearly,

$$\mu(x+y) \geq \mu(x) \wedge \mu(y) \Rightarrow \mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$$

$$\mu(\alpha x) \geq \mu(x) \Rightarrow \mu(\alpha x) \geq \mu(x) \wedge 0.5$$

$$\mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \Rightarrow \mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5.$$

Proposition 3.3. A non-empty set I of L is a 3-Lie subalgebra of L if and only if χ_I is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra of L.

Proof. Assume that I is an 3-Lie subalgebra of L. Then χ_I is a fuzzy 3-Lie algebra in the sense of Proposition 2.4 and so it is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra.

Conversely, assume that χ_I is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra of L. Then for every $x, y, z \in I$, we have:

$$\chi_I(x+y) \geq \chi_I(x) \wedge \chi_I(y) \wedge 0.5 = 0.5,$$

$$\chi_I(\alpha x) \geq \chi_I(x) \wedge 0.5 = 0.5,$$

$$\chi_I([x,y,z]) \geq \chi_I(x) \wedge \chi_I(y) \wedge \chi_I(z) \wedge 0.5 = 0.5.$$

So $x+y \in I, \alpha x \in I, [x,y,z] \in I$. Therefore I is a 3-Lie subalgebra of L.

Now, we characterize $(\in, \in \vee q)$-fuzzy 3-Lie algebra by their level subalgebra.

Theorem 3.4. Let μ be an $(\in, \in \vee q)$-fuzzy 3-Lie algebra of L. Then for all $0 < t \leq 0.5$, $[\mu]_t$ is a non-empty subalgebra of L. Conversely, if μ is a fuzzy subset of L such that $[\mu]_t \neq \emptyset$ is a 3- Lie subalgebra of L for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra of L.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy 3-Lie algebra of L and $0 < t \leq 0.5$. Let $x, y, z \in [\mu]_t$. Then $\mu(x) \geq t, \mu(y) \geq t, \mu(z) \geq t$. Now we have:

$$\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t,$$

$$\mu(\alpha x) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5 = t,$$

$$\mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5 \geq t \wedge 0.5 = t.$$

So $x+y \in [\mu]_t, \alpha x \in [\mu]_t, [x,y,z] \in [\mu]_t$. Therefore, $[\mu]_t$ is a 3-Lie subalgebra of L.

Conversely, let μ be a fuzzy subset of L such that $[\mu]_t \neq \emptyset$ be a 3-Lie subalgebra of L for all $0 < t \leq 0.5$. For every $x, y, z \in L, \alpha \in F$, we can write $\mu(x) \wedge \mu(y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = k_0$, then $x \in [\mu]_{k_0}$ and $y \in [\mu]_{k_0}$. So $x+y \in [\mu]_{k_0}$. Therefore, $\mu(x+y) \geq k_0 = \mu(x) \wedge \mu(y) \wedge 0.5$. Similarly, we have $\mu(\alpha x) \geq \mu(x) \wedge 0.5$ and $\mu([x,y,z]) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5$. Hence, μ is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra of L.

Naturally, a corresponding result should be considered when $[\mu]_t \neq \emptyset$ is an 3-Lie subalgebra of L for all $t \in (0.5, 1]$.

Theorem 3.5. Let μ be a fuzzy subset of a 3-Lie algebra L. Then $[\mu]_t \neq \emptyset$ is a 3- Lie subalgebra of L for all $t \in (0.5, 1]$ if and only if:

(1) $\mu(x+y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ for all $x, y \in L$,

(2) $\mu(\alpha x) \vee 0.5 \geq \mu(x)$ for all $x \in L, \alpha \in F$,

(3) $\mu([x,y,z]) \vee 0.5 \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$.

Proof. Suppose that $[\mu]_t \neq \emptyset$ is a 3-Lie subalgebra of L.

(1): Suppose that for some $x, y \in L$, $\mu(x+y) \vee 0.5 < \mu(x) \wedge \mu(y) = t$, then $t \in (0.5, 1]$, $\mu(x+y) < t, x \in [\mu]_t$ and $y \in [\mu]_t$. Since $x, y \in [\mu]_t$ and $[\mu]_t$ is a 3-Lie subalgebra, so $x+y \in [\mu]_t$ or $\mu(x+y) \geq t$, which is a contradiction with $\mu(x+y) < t$. Hence (1) holds.

(2): Suppose that for some $x \in L$, $\mu(\alpha x) \vee 0.5 < \mu(x) = t$, then $t \in (0.5, 1]$, $\mu(\alpha x) < t$ and $x \in [\mu]_t$. Since $x \in [\mu]_t$, we get $\alpha x \in [\mu]_t$ or $\mu(\alpha x) \geq t$, which is a contradiction. Hence (2) holds.

(3): Suppose that for some $x, y, z \in L$, $\mu([x,y,z]) \vee 0.5 < \mu(x) \wedge \mu(y) \wedge \mu(z) = t$, then $t \in (0.5, 1]$, $\mu([x,y,z]) < t$, $x \in [\mu]_t$, $y \in [\mu]_t$, $z \in [\mu]_t$,. Since $x, y, z \in [\mu]_t$ and $[\mu]_t$ is a 3-Lie subalgebra, so $[x,y,z] \in [\mu]_t$ or $\mu([x,y,z]) \geq t$, which is a contradiction with $\mu([x,y,z]) < t$. Hence (3) holds.

Now, suppose that conditions (1),(2),(3) hold. We show that $[\mu]_t$ is a 3-Lie subalgebra of L. Assume that $t \in (0.5, 1]$, $x, y, z \in [\mu]_t$. Then

$$0.5 < t \leq \mu(x) \wedge \mu(y) \leq \mu(x+y) \vee 0.5 \Rightarrow \mu(x+y) \geq t,$$

$$0.5 < t \leq \mu(x) \leq \mu(\alpha x) \vee 0.5 \Rightarrow \mu(\alpha x) \geq t,$$

$$0.5 < t \leq \mu(x) \wedge \mu(y) \wedge \mu(z) \leq \mu([x,y,z]) \vee 0.5 \Rightarrow \mu([x,y,z]) \geq t,$$

and so $x+y \in [\mu]_t$, $\alpha x \in [\mu]_t$ and $[x,y,z] \in [\mu]_t$. Therefore, $[\mu]_t$ is a 3-Lie subalgebra of L.

Definition 3.6. Let $\alpha, \beta \in [0,1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of a 3-Lie algebra L. Then μ is called a fuzzy 3-Lie algebra with thresholds of L, if for all $x, y, z \in L$ and $\gamma \in F$,

(1) $\mu(x+y) \vee \alpha \geq \mu(x) \wedge \mu(y) \wedge \beta$,

(2) $\mu(\gamma x) \vee \alpha \geq \mu(x) \wedge \beta$,

(3) $\mu([x,y,z]) \vee \alpha \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \beta$.

If μ is a fuzzy 3-Lie algebra with thresholds of L, then we can conclude that μ is an ordinary fuzzy 3-Lie algebra when $\alpha = 0$, $\beta = 1$; and μ is an $(\in, \in \vee q)$-fuzzy 3-Lie algebra when $\alpha = 0$, $\beta = 0.5$.

Now, we characterize fuzzy 3-Lie algebra with thresholds by their level 3-Lie algebras.

Theorem 3.7. A fuzzy subset μ of a 3-Lie algebra L is a fuzzy 3-Lie algebra with thresholds of L if and only if $[\mu]_t \neq \emptyset$ is a 3-Lie algebra of L for all $t \in (\alpha, \beta]$.

Proof. Let μ be a 3-Lie algebra with thresholds of L. Let $x, y, z \in [\mu]_t$, $\gamma \in F$, then $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(z) \geq t$. Now we have:

$$\mu(x+y) \vee \alpha \geq \mu(x) \wedge \mu(y) \wedge \beta \geq t \wedge \beta = t \Rightarrow \mu(x+y) \geq t,$$

$$\mu(\gamma x) \vee \alpha \geq \mu(x) \wedge \beta \geq t \wedge \beta = t \Rightarrow \mu(\gamma x) \geq t,$$

$$\mu([x,y,z]) \vee \alpha \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \beta \geq t \wedge \beta = t \Rightarrow \mu([x,y,z]) \geq t,$$

and so $x + y \in [\mu]_t$, $\gamma x \in [\mu]_t$ and $[x,y,z] \in [\mu]_t$. Therefore $[\mu]_t$ is a 3-Lie algebra of L.

Conversely, let μ be a fuzzy subset of L such that $[\mu]_t \neq \emptyset$ be a subalgebra of L for all $\alpha < t \leq \beta$. We show that μ is a 3-Lie algebra with thresholds of L or

(1) $\mu(x+y) \vee \alpha \geq \mu(x) \wedge \mu(y) \wedge \beta$,

(2) $\mu(\gamma x) \vee \alpha \geq \mu(x) \wedge \beta$,

(3) $\mu([x,y,z]) \vee \alpha \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \beta$.

for all $x, y, z \in L, \gamma \in F$.

(1): Suppose that for some $x, y \in L$, $\mu(x+y) \vee \alpha < \mu(x) \wedge \mu(y) \wedge \beta = t$, then $t \in (\alpha, \beta]$, $\mu(x+y) < t$, $x \in [\mu]_t$ and $y \in [\mu]_t$. Since $x, y \in [\mu]_t$ and $[\mu]_t$ is a

3-Lie subalgebra, so $x+y \in [\mu]_t$ or $\mu(x+y) \geq t$, which is a contradiction with $\mu(x+y) < t$. Hence (1) holds.

(2): Suppose that for some $x \in L$, $\mu(\gamma x) \vee \alpha < \mu(x) \wedge \beta = t$, then $t \in (\alpha, \beta]$, $\mu(\gamma x) < t$ and $x \in [\mu]_t$, which is a contradiction with $\mu(\gamma x) < t$. Hence (2) holds.

(3): Suppose that for some $x, y, z \in L$, $\mu([x,y,z]) \vee \alpha < \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \beta = t$, then $t \in (\alpha, \beta]$, $\mu([x,y,z]) < t$ and $x, y, z \in [\mu]_t$. Since $x, y, z \in [\mu]_t$ and $[\mu]_t$ is a 3-Lie subalgebra, so $[x,y,z] \in [\mu]_t$ or $\mu([x,y,z]) \geq t$, which is a contradiction with $\mu([x,y,z]) < t$. Hence (3) holds.

Therefore μ is a 3-Lie algebra with thresholds of L.

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ADMISSIBLE SUPERHARMONIC FUNCTIONS IN RIEMANNIAN SPACES

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Abstract

The existence of biharmonic Green functions in a Riemannian manifold or in Riemann surface Ω is related to the Riesz representation of certain δ-superharmonic functions Ω.[1]

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1. Introduction

M. Nakai and L. Sario [5] , [6] show that certain biharmonic functions u with Dirichlet finite Laplacian in a Riemannian manifold Ω have a Riesz representation as the sum of a harmonic function and the Green potential of Δu where $\Delta = -\sum_{i,j} g^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$ is the Laplace-Beltrami operator. These results have led us to investigate the class F of locally integrable functions s on Ω which have a Riesz representation. We prove that a hyperbolic Riemannian manifold Ω has biharmonic Green functions if and only if F contains a function s for which Δs is superharmonic positive in Ω; a special case of interest is of course when $\Delta s = 1$.

In [2] are given more general representations for biharmonic and bisubharmonic functions in $\mathbb{R}^n$, $n \geq 2$, following the Nevanlinna theory for meromorphic functions in complex plane. Here we use different methods in Ω, which can be a Riemannian manifold or a Riemann surface, to obtain a Riesz-type representation for some of these functions. These methods require a minimal rule for Δ as a differential operator on Ω.

Consequently, we can deal with biharmonic and bisuperharmonic functions on Ω whether Ω is a Riemannian manifold or a Riemann surface and whether Ω is hyperbolic or parabolic. Note that in the context of Nakai-Sario papers, biharmonicity is not meaningful in an abstract Riemann surface since the Laplacian is not invariant under a parametric change (P.6[9]); also there only biharmonic functions in a hyperbolic Riemannian manifold are considered.

2. Preliminaries Let Ω stand for a Riemannian manifold on a Riemann surface; and dx stands for the value or the surface area element. Ω is assumed to be non compact. As usual, Ω is said to be hyperbolic if there exists a Green function (or equivalently a potential) in Ω; otherwise Ω is parabolic. Using an approximation lemma (for Riemann surfaces [7])

Lemma 2.1 *let $\mu \geq 0$ be a Radon measure on an open set $\omega \subset \Omega$. Then, there exists a superharmonic functions s in $\omega$with associated measure μ in a local Riesz representation. The converse of this lemma is valid in the form: For any superharmonic function s in an open set $\omega \subset \Omega$, there is an associated measure $\mu \geq 0$ such that in a parametric ball B, $\bar{B} \subset \omega$, $s(x) = h(x) + \int_B G_B(x,y) d\mu(y)$ where $h(x)$ is a harmonic function and G_B is the Green function in B.*

Let us denote this relation between s and μin ω by the equation $Ls = -\mu$ in ω. Thus, if s denotes the superharmonic functions and M_+ the Radon measure ≥ 0 in Ω, then $L : S \to M_+$ is a linear, onto map which is not one − to − one . Suppose μ is a singed measure on Ω. Then we have a δ-sperharmonic function u in Ω such that $Lu = -\mu$. We say that u is generated by μ. Given a locally dx-integrable function f in Ω, let μ be the signed measure defined by $d\mu(x) = f(x)dx$. If u is the δ-superharmonic function generated by μ, we write simply $Lu = -f$ in Ω.

3. Admissible densities

Let Ω be a Riemannian manifold with a countable base or a Riemann surface, hyperbolic or parabolic. A superharmonic function s in Ω is said to be admissible if s has a harmonic minarant outside a compact set.

u is said to be an admissible δ-superharmonic functions in Ω if u is the difference of two admissible superharmonic functions in Ω.

Remarks:

1) If Ω is hyperbolic, $u$is an admissible δ-superharmonic function in Ω if and only if u is the difference of two potentials up to an additive harmonic function, that is if and only if u has the Riesz representation in the sence of [9]. Chapter VII.

2) Let u be an admissible δ-superharmonic function in a hyperbolic Ω. Then u has a subharmonic minarant; if $\{\Omega_n\}$ is a regular exhaustion of Ω and if $H_u^{\Omega_n}$ denote the Dirichlet solution in Ω with boundary values u, then $H_u^{\Omega_n}$ converges locally uniformly to a harmonic function. But these two properties are not sufficient to characterize admissible δ-superharmonic functions.

3) An example of a bounded biharmonic function that is not an admissible δ-superharmonic function: let $\Omega = \mathbb{R}^3 \backslash \{0\}$. For $x \in \Omega$, let $(x_1, x_2, x_3)$and (r, θ, φ)be the Cartesian and the polar coordinates. Let $u(x) = \frac{x_3}{r}$ in Ω. Then u is bounded harmonic function for which $|\Delta u| = \frac{2}{r^2}|\cos\theta|$. Hence if u is an admissible δ-superharmonic function, we should have $\int_\Omega \frac{1}{|x_0 - x|}\frac{2}{r^2}|\cos\theta| r^2 dr\, d\omega$ finite where $d\omega$ is the surface area of the unit sphere, $|x_0| < 1$ and $|x| = r$. This means that $\int_1^\infty \frac{1}{r+1} dr$ is finite, which is not true.

Theorem 3.1 *Let s be a superharmonic function with associated measure μ in a parabolic Ω. Then s admissible if and only if $\mu(\Omega)$ is finite.*

Proof: i) Let $\mu(\Omega)$ be finite , and suppose Ω is hyperbolic with $G(x,y)$ as the Green potential. Then $p(x) = \int_\Omega G(x,y)d\mu(y)$ is a well defined potential in Ω . For, if B is a parametric ball $p_1(x) = \int_{\bar{B}} G(x,y)d\mu(y)$ is well defined since $\bar{B}$ is compact. Let $p_2(x_0) < \infty$, $x_0 \in B$. Now $G(x_0,y)$, as a function of y is a potential in Ω with point support $x_0 \in B$. Hence $G(x_0,y) = D(G(x_0,y))$ in $\Omega\backslash\bar{B}$, where $D(G(x_0,y))$ is the Dirichlet solution in $\Omega\backslash\bar{B}$ with boundary values $G(x_0,y)$ for $y \in \partial B$ and 0 at the point at infinity. Hence if $M = \max_{y\in\partial B} G(x_0,y)$, $G(x_0,y) \leq M$ for $y \in \Omega\backslash B$. Hence $\int_{\Omega\backslash B} G(x_0,y)d\mu(y) \leq M \int_{\Omega\backslash B} d\mu(y) < \infty$.
Thus. $\int_\Omega G(x_0,y)d\mu(y) = \int_{\bar{B}} G(x_0,y)d\mu(y) + \int_{\Omega\backslash\bar{B}} G(x_0,y)d\mu(y) < \infty$. Consequently, $p(x) = \int_\Omega G(x,y)d\mu(y)$ is well defined as a potential in Ω if it is hyperbolic.

Now, suppose Ω is parabolic as in statement of the theorem and that $\mu(\Omega)$ is finite. Let B be a parametric ball in Ω. Then $\Omega\backslash\bar{B}$ is hyperbolic. Hence, by the above argument there exists a positive superharmonic function $p(x)$ in $\Omega\backslash\bar{B}$ such that the measure associated with $p(x)$ is μ in $\Omega\backslash\bar{B}$. Since by the assumption, $s(x)$ is a superharmonic function in Ω with associated measure μ, we have $s(x) = p(x)$+a harmonic function in $\Omega\backslash\bar{B}$, which implies that $s(x)$ has a harmonic minorant is $\Omega\backslash\bar{B}$, that is , s is admissible.

ii) Conversely, let s be an admissible superharmonic function in a parabolic Ω with associated measure μ; then $\mu(\Omega)$ should be finite, this follows from Theorem 4.1 [1] proved in a more general context.

Definition 3.2 *A signed measure μ in Ω is said to be an admissible density, written $\mu \in \underline{A}$ if μ generates an admissible δ-superarmonic function u, that is $Lu = -\mu$.*

A locally, $dx-$integrable function $f \in A$if and only if the signed measure μ defined by $d\mu(x) = f(x)dx$ belongs to $\underline{A}$.

Lemma 3.3 *Let μ and ν be two positive measures such that $\nu \leq \mu$ and $\mu \in \underline{A}$. Then $\nu \in \underline{A}$.*

Proof: Let $\lambda = \mu - \nu$. By lemma 2.1, there exist supperharmonic functions u, $v$and s such that $Lu = -\mu$, $Lv = \nu$ and $Ls = -\lambda$. Here u is admissible since $\mu \in \underline{A}$. Then $s + v = u$+a harmonic function in Ω. Since u has a harmonic minorant outside a compact set K, $v$has a subharmonic minorant in $\Omega\backslash K$. This implies that $\nu \in \underline{A}$.

Lemma 3.4 *A signed measure $\mu \in \underline{A}$ if and only if $|\mu| \in \underline{A}$.*

Proof: 1) Let $\mu \in \underline{A}$. Let $u = s_1 - s_2$ be an admissible δ-superharmonic function such that $Lu = -\mu$. Then $-(\mu^+ - \mu^-) = L(u) = Ls_1 - Ls_2 = -\mu_1 + \mu_2$ where μ_1 and μ_2 are positive measures. Hence (by country-region place Jordan decomposition) $\mu^+ \leq \mu_1$ and $\mu^- \leq \mu_2$ and consequently by Lemma 3.3 there exist admissible supperharmonic functions u_1 and u_2 such that $Lu_1 = -\mu^+$ and $Lu_2 = -\mu_2^-$. Hence $L(u_1 + u_2) = -|\mu|$ which means that $|\mu| \in \underline{A}$.

2) Let $|\mu| \in \underline{A}$. Since μ^+ and $\bar{\mu}$ are majorized by $|\mu|$, by lemma 3.3 there exist admissible supperharmonic functions s_1 and s_2 in Ω such that $Ls_1 = -\mu^+$ and $Ls_2 = -\bar{\mu}$. Consequently, $u = s_1 - s_2$ is an admissible δ-supperharmonic function such that $Lu = -\mu$, that is $\mu \in \underline{A}$.

Theorem 3.5 *Let f and s be two locally $dx-$integrable functions in Ω. Suppose $s \in \underline{A}$ and $|f| \leq |s|$ outside a compact set. Then $f \in \underline{A}$.*

Proof: Let $|f| \leq |s|$ outside the compact set K in Ω. Let ω be a relatively compact domain containing K.

Let φ_1 be a continuous function in Ω. $o \leq \varphi_1 \leq 1$, $\varphi_1 = 0$ for K, $\varphi_1 = 1$ on ω^c.

Let φ_2 be a continuous function in Ω with compact support, $o \leq \varphi_2 \leq 1$ $\varphi_2 = 1$ on ω.

Let $\varphi_1 = f\varphi_1$ and $f_2 = f\varphi_2$. Then $|f| \leq |f_1| + |f_2|$ on Ω. Since $|f_1| \leq |s|$ in Ω and $s \in \underline{A}$, by Lemmas 3.3 and 3.4 $|f_1| \in \underline{A}$.

Since $|f_2|$ is locally $dx-$integrable function and has compact support in Ω, $|f_2| \in \underline{A}$.

Hence $|f_1| + |f_2| \in \underline{A}$ and since $|f| \leq |f_1| + |f_2|$ in Ω, $|f|$ and hence f in $\underline{A}$.

Admissible densities in $\mathbb{R}^n$: Let f be a locally $dx-$integrable (that is, Lebesgue integrable) function in $\mathbb{R}^n$, $n \geq 2$. Suppose $f$is an admissible density function. Then $|f| \in \underline{A}$ and if $Lu = -|f|$, u is an admissible superharmonic function in $\mathbb{R}^n$.

This means when $n = 2$, that $\int_{\mathbb{R}^n} |f(x)|\, dx < \infty$(Theorem 3.1); and when $n \geq 3$, u is a potential up to an additive harmonic function, which implies that $\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} |f(x)|\, dx$ is finite for some y in $\mathbb{R}^n$.

Thus, we can say that a locally Lebesgue integrable function $f(x)$ in $\mathbb{R}^n$, $n \geq 2$ is an admissible density function if and only if $\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} |f(x)|\, dx$ is finite for some $y \in \mathbb{R}^n, n \geq 2$.

Proposition 3.6 *Every admissible biharmonic function in* $\mathbb{R}^n$, $n \geq 2$ *is harmonic.*

Proof: Let $b(x)$ be an admissible biharmonic function in $\mathbb{R}^n$, $n \geq 2$. This means that if $Lb = -h$, then the harmonic function h is an admissible density function in $\mathbb{R}^n$. Hence, given any $x_0 \in \mathbb{R}^n$, $\int_{|x-x_0|} \frac{1}{|x-x_0|^{n-2}} |h(x)|\, dx$ should be finite. That is, if $r = |x - x_0|$, $\int_{r=1}^{\infty} \int_{s_r} \frac{1}{r^{n-2}} |h(r,\omega)|\, r^{n-1} dr d\omega$ is finite, where $d\omega$ is the surface area of the unit sphere S_1. Since $|h(x_0)| \leq \frac{1}{A_n} \int_{s_r} |h(r,\omega)|\, d\omega$, where $r > 1$ and A_n is the surface area of S_1 in $\mathbb{R}^n$, we should have $\int_1^{\infty} |h(x_0)|\, r dr$ finite. Hence $h(x_0) = 0$. x_0 being arbitary, $h \equiv 0$ and b is harmonic in $\mathbb{R}^n$.

4. Admissible Riemannian space

Let Ω be a Riemannian manifold or a Riemann surface, hyperbolic or parabolic.

Definition 4.1 Ω *is said to be an admissible space if there exists a superharmonic function* $s > 0$ *that is an admissible density,* $s \in \underline{A}$. Ω *is said to be strongly admissible if* $1 \in \underline{A}$.

Remarks :

1) When $n \geq 5$, $\mathbb{R}^n$ is an admissible space but not strongly admissible. For $u(x) = |x|^2$is biharmonic, Lu is a constant but u is not admissible (Remark (2) is the beginning of section 3)

2) If Ω is parabolic, Ω is strongly admissible if and only if it is admissible. An example of a parabolic Ω that is admissible: The Poincaré 3-ball $\Omega = B_1^3$ is parabolic and $\int_\Omega dx$ is finite (pp.40-43 [9]).

Definition 4.2 *A* δ*-superharmonic function* u *in* Ω *said to be a bisuperharmonic function if* $Lu = -s$ *where* s *is a superharmonic function in* Ω. *If* $\underline{q}$ *and* p *are two potentials in* Ω *such that* $L\underline{q} = -p$, *then* $\underline{q}$ *is called bipotential.*

Given $y \in \Omega$, let $\Im$ be an increasingly filtered family of relatively compact domains ω in Ω such that $y \in \omega$ and $\Omega = \bigcup_{\Im} \omega$. For $w \in \Im$, let G_ω be the Green potential in ω with pole $\{y\}$. There exists a unique potential $\underline{q}_\omega$ in ωsuch that $L\underline{q}_\omega = -G_\omega$; also $\{\underline{q}_\omega\}$ is increasing. Now the Sario Criterion [8] states that when Ω is a Riemannian manifold and sup $\underline{q}_\omega \not\equiv \infty$, the biharmonic Green function can be defined in Ω. It is proved in [3] that the Sario Criterion is satisfied if and only if there exists a bipotential in Ω.

In view of section 2, this proof is carried over even when Ω is a Riemann surface. We state this as the following lemma.

Lemma 4.3 *Let Ω be a Riemannian manifold or a Riemann surface. The biharmonic Green function exists in Ω if and only if there exist a bipotential in Ω.*

Theorem 4.4 *Let Ω be Riemannian manifold or a Riemann surface and hyperbolic. Ω has the biharmonic Green function if and only if Ω is admissible.*

Proof: 1) Suppose Ω is a admissible. That is there exists a superharmonic function $s > 0$ that is an admissible density. Let $Lu = -s$. Then u is an admissible superharmonic function since Ω is hyperbolic, u has a harmonic minorant in Ω. Since $s > 0$ is superharmonic, write $s = p_1 + h$, where p_1 is a potential and h is harmonic ≥ 0.

a) Suppose $p_1 \equiv 0$. Take the family of all superharmonic functions $v > 0$ in Ω such that $v \geq h$ on a fixed parametric ball B. Then $p_0 = \inf v = \hat{R}_h^B$ is a potential in Ω, $p_0 = h$ on B, p_0 is harmonic in $\Omega \backslash B$ and $p_0 \leq h$ in Ω. Thus $o < p_0 \leq h = s$ is Ω.

b) If $p_1 \not\equiv 0$, $0 < p_1 \leq s$ in Ω . Thus is both the cases , there exists a potential p in Ω such that $o < p \leq s$. Since $s \in \underline{A}$, $p \in \underline{A}$ (Theorem 3.5). Let $Lf = -p$. Then f is an admissible superharmonic function in the hyperbolic space Ω. Let H be greatest harmonic minorant of f in Ω and write $\underline{q} = f - H$. Then $\underline{q}$is a potential in Ω and $L\underline{q} = Lf = -p$. Hence by lemma 4.3, the biharmonic Green function exists in Ω.

2) Conversely, if Ω has the biharmonic function, then by lemma 4.3 there exist potentials $\underline{q}$ and p such that $L\underline{q} = -p$. Thus $p \in \underline{A}$ and Ω is admissible.

Theorem 4.5 *Let Ω be a Riemannian manifold or a Riemann surface, hyperbolic or parabolic. Then Ω is strongly admissible if and only if every locally $dx-$integrable function $f(x)$, that is bounded outside a compact set is an admissible density function, $f \in \underline{A}$.*

Proof: By definition, Ω is strongly admissible if and only if $1 \in \underline{A}$. Hence this theorem is a consequence of Theorem 3.5.

Remark: In a Riemannian manifold, according to [9], $s \in H^2C_\Delta$ if s is biharmonic and $\Delta s \neq 0$ is a bounded and Dirichlet finite function; and $u \in H^2DC_\Delta$ if u is a Dirichlet finite nonharmonic biharmonic function for which Δu is a bounded and Dirichlet finite function. It is shown (Theorem

1.5, p.251 [9]) that if $\Omega \in \tilde{O}^N_{QP}$ (that is $\Omega$is a strongly admissible hyperbolic Riemannian manifold), then H^2DC_Δ is one of the families that has the Riesz representation (that is, every function in H^2DC_Δ is admissible).

Such results for the families like H^2C_Δ, H^2DC_Δ can be got as corollaries of Theorem 4.5.

Proposition 4.6 *Let Ω be a strongly admissible space, let f be a locally $dx-$integrable function such that $|f|^p \in \underline{A}$, $1 \leq p \leq \infty$. Then $f \in \underline{A}$.*

Proof: If $p = 1$, the proposition follows from Lemma 3.4 even if Ω is not admissible; if $p = \infty$, the assumptions imply that $|f| \leq 1 \quad a.e$, and hence $f \in \underline{A}$ since Ω is strongly admissible (Theorem 3.5).

When $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$. Let $K(x,y) = G(x,y)$ be the Green function if Ω is hyperbolic and $K(x,y) = 1$ if Ω is parabolic. Since Ω is strongly admissible and since $|f|^p \in \underline{A}$, we can choose some $x_0 \in \Omega$ such that $\int_\Omega K(x_0,y)dy < \infty$ and $\int_\Omega K(x_0,y)\,|f(y)|^p\,dy < \infty$. Hence,

$$\int_\Omega K(x_0,y)\,|f(y)| \leq \left[\int_\Omega K(x_0,y)\,|f(y)|^p\,dy\right]^{\frac{1}{p}} \left[\int_\Omega K(x_0,y)dy\right]^{\frac{1}{q}} < \infty.$$

Hence $f \in \underline{A}$.

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ON THE COZERO DIVISOR GRAPHS OF COMMUTATIVE SEMIGROUPS

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Abstract

In this paper, we define and study the cozero divisor graph of a commutative semigroup S, denoted by $\Gamma'(S)$. Also we investigate some graph theoretical properties of $\Gamma'(S)$ including its planarity, independence number and clique number of the graph.

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1. Introduction

The zero divisor graph of a commutative ring was introduced by Beck [11], but this work was mostly concerned with coloring of rings. In [15], the study of the zero divisor graphs of semigroups was begun. There are several types of graphs associated to a semigroup, see for example [7], [9], [17], [18] and [19]. Also in [3], Afkhami and Khashyarmanesh introduced the cozero divisor graph of a commmutative ring R, which is denoted by $\Gamma'(R)$, that examines relationships between principal ideals.

In this paper, we define and investigate the cozero divisor graph of a commutative semigroup S with multiplicative operation, denoted by $\Gamma'(S)$. The cozero divisor graph $\Gamma'(S)$ is an undirected graph whose vertex set consists of all the non-unit elements of S, and two distinct vertices a and b are adjacent if and only if $a \notin (b)$ and $b \notin (a)$, where, for an element c in S, (c) is the ideal Sc. We denote the vertex set of $\Gamma'(S)$ by $W(S)$. In this paper we study some graph theoretical properties of $\Gamma'(S)$, namely planarity, clique number, choromatic number, Hamiltonian cycle and Eulerian cycle.

A simple graph is a pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of G, respectively. In a graph G, the *distance* between two distinct vertices x and y, denoted by $d_G(x, y)$, is the length of the shortest path connecting x and y, if such a path exists, otherwise, we set $\mathrm{d}_G(x, y) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}_G(x, y) \ : \ x \text{ and } y \text{ are distinct vertices of } G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, $\mathrm{gr}(G) := \infty$. The *degree* of a vertex x is the number of the edges of the graph G incident with x. A *clique* of G is a complete subgraph of it and the number of vertices in a largest clique is called the *clique number* of G and is denoted by $\omega(G)$. Also, for two distinct vertices x and y in G, the notation $x - y$ means that x and y are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. Also, we say that G is an *empty graph* if no two distinct vertices of G are adjacent, and G is a *null graph* if it contains no vertices. A subset $C \subseteq V$ is called an *independent set* if the subgraph of G which is induced by C has no edges. An *independence number*, which is denoted by $\alpha(G)$, is the cardinality of the largest independente set. A *dominating set* for a graph G is a subset D of V such that every vertex not in D is adjacent to at least one vertex of D. The *domination number*, denoted by $\lambda(G)$, is the number of vertices in a smallest dominating set of G. A *cutset* in a graph G is a set of vertices whose deletion increases the number of connected components of G. The *vertex connectivity* of a connected graph G is the minimum number of vertices in a cutset, and is denoted by $K_0(G)$. Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$. The vertex *Laplacian* matrix of G, denoted by $L(G)$, is the $n \times n$ matrix defined as follows. The rows and columns of $L(G)$ are indexed by $V(G)$. If $i \neq j$, then the (i, j)-entry of $L(G)$ is -1 if i and j are adjacent, and it is 0 if i and j are not adjacent. The (i, i)-entry of $L(G)$ is d_i, the degree of the vertex i, for $i = 1, 2, \ldots, n$. A *tree* is an undirected graph in which any two vertices are connected by exactly one simple path. A *spanning tree* of a connected, undirected graph G is a tree

that contains all vertices of G. The notation $\kappa(G)$ is denoted for the number of spanning trees of a graph G. A graph G is *Eulerian* if there exists a closed trail containing every edge. By [13, Theorem 4.1], a connected non-empty graph is Eulerian if and only if the degrees of all vertices are even. A *Hamiltonian cycle* in a graph G is a cycle that contains every vertex of G. Moreover G is called *Hamiltonian* if it contains a Hamiltonian cycle. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets, so that no edges has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{n,m}$. A graph is called *planar* if it can be drawn in the plane without any edge crossing. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [13, p. 153]). The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *split graph* is a graph in which the vertices can be partitioned in to a clique and an independent set. For any vertex x of a connected graph G, the *eccentricity* of x, denoted by $\varepsilon(x)$, is the maximum of the distances from x to the other vertices of G, and the minimum value of the eccentricity is the *radius* of G, which is denoted by $r(G)$. The *complement* of a graph G, which is denoted by $\overline{G}$, is a graph whose vertices are the same as the vertices of G, and we have $\{x, y\} \in E(\overline{G})$ if and only if $\{x, y\} \notin E(G)$. The *line graph* $L(G)$ of a graph G, is the graph with $V(L(G)) = E(G)$ and $\{e_1, e_2\} \in E(L(G))$ if and only if e_1 and e_2 are incident to a common vertex in G. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(G)$ are indexed by $V(G)$. If $i \neq j$, then the (i, j)-entry is 0 for nonadjacent vertices i and j, and (i, j)-entry is 1 for adjacent vertices i and j. The (i, i)-entry of $A(G)$ is 0 for $i = 1, \ldots, n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A(G)$. Recall that the *energy* of G is defined as $\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$. We denote S_g for the surface formed by a connected sum of g tori and the number g is called the *genus* of the surface S_g. A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. The notation $\gamma(G)$ is denoted for the genus of a graph G. Clearly a graph G is planar if $\gamma(G) = 0$, and it is *toroidal* if $\gamma(G) = 1$ (see [21, 22]).

2. The Cozero divisor graph of a commutative semigroup

Throughout this paper S is a commutative semigroup. Recall that a non-empty subset I of a semigroup S is called an ideal if $SI \subseteq I$. Clearly $Sa = \{sa : s \in S\}$ is an ideal of S which is called the principal ideal generated by a, and we denote it by (a). Also an ideal M of S is called Maximal if $M \subsetneqq S$ and there does not exist an ideal M_1 of S such that $M \subsetneqq M_1 \subsetneqq S$. We denote the set of all maximal ideals of S by A, and we set $M^* := \bigcap_{M \in A} M$. Also, recall that a non-trivial ideal Q of a semigroup S is said to be prime if $IJ \subset Q$ implies that $I \subset Q$ or $J \subset Q$, where I and J are ideals of S. We denote the set of all prime ideals of S by B, and we set $Q^* := \bigcap_{Q \in B} Q$.

Remark 2.1. *Let I be a proper ideal of S, $T = S \backslash I$ and $a \in T$. By [20], I is a maximal ideal of S if and only if either T is the set $\{a\}$, where $a^2 \in I$, or $T = \{x \in S : (x) = (a)\}$, where $a \in Sa$.*

For the rest of the paper we assume that S has at least two maximal ideals. In the following theorem, we show that $\Gamma'(S)\backslash M^*$ is connected.

Theorem 2.2. *The graph $\Gamma'(S)\backslash M^*$ is connected and* $\operatorname{diam}(\Gamma'(S)\backslash M^*) \leq 2$.

Proof. Let x and y be arbitrary distinct vertices in $\Gamma'(S)\backslash M^*$ such that x is not adjacent to y. Then $x \in (y)$ or $y \in (x)$. Without loss of generality, we may assume that $x = ry$ for some $r \in S$. Suppose that $A = \{M_\alpha : \alpha \in \Lambda\}$ and $y \in M_\alpha$, for some $\alpha \in \Lambda$. Then $x \in M_\alpha$. Also, since $x \notin M^*$, there exists $\beta \in \Lambda$ such that $x \notin M_\beta$. So $y \notin M_\beta$. By Remark 2.1, we may assume that $M_\alpha = S - T$ and $M_\beta = S - T'$ for some sets T and T'. If $|T| = |T'| = 1$, then we have $x = b = y$, where $b \in T'$, which is impossible. Hence we have $|T| > 1$ or $|T'| > 1$. Suppose that $a \in T$. Then clearly $a \notin (x)$ and $a \notin (y)$. Suppose that $x \in (a)$. In view of the proof of Theorem 2 in [20], we have $M_\alpha \cup M_\beta = S$. Hence we have $x \in M_\beta$, which is a contradiction. Therefore $x \notin (a)$. Similarly $y \notin (a)$. Thus we have the path $x - a - y$.

By above discussion, we have that $\operatorname{diam}(\Gamma'(S)\backslash M^*) \leq 2$. □

In [20], it is proved that if S is a semigroup with $S = S^2$, $A \neq \emptyset$ and $Q^* \neq \emptyset$, then $Q^* \subseteq M^*$. Also if S is a semigroup containing at least one maximal ideal, then every prime ideal of S is a maximal ideal if and only if $M^* \subseteq Q^*$. Moreover, if S is a semigroup with $S = S^2$ and containing at least one maximal ideal, then $A = B$ if and only if $M^* = Q^*$. Now we have the following corollary.

Corollary 2.3. *Suppose that S is a semigroup with $S = S^2$ and every prime ideal of S is maximal. Then the graph $\Gamma'(S)\backslash Q^*$ is connected.*

In the following theorem, we study the case that $\Gamma'(S)$ is complete.

Theorem 2.4. *Assume that S is a semigroup with the identity element 1 and $M^* = \emptyset$. Then the following statements are equivalent:*

(i) *For each maximal ideal $M \in A$, we have $M = S - \{a\}$, where $a^2 \in M$.*
(ii) *The graph $\Gamma'(S)$ is complete.*
(iii) *For all vertices $a \in W(S)$, we have $|(a)| = 1$.*

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose that $M = S - \{a\}$, where $a^2 \in M$, for all $M \in A$. Also, suppose that x and y are arbitrary distinct nonadjacent vertices in $\Gamma'(S)$. Then, in view of Theorem 2.2, there exists a maximal ideal $M_\alpha = S - \{a_\alpha\}$ in A such that $x, y \notin M_\alpha$. Hence we have $x = a_\alpha = y$, which is a contradiction. So $\Gamma'(S)$ is a complete graph.

(ii) $\Rightarrow$ *(iii)* Suppose that $\Gamma'(S)$ is complete. If there is a vertex x in $\Gamma'(S)$ with $|(x)| \geq 2$, then there exists a non-zero element y such that $y \in (x)$ and $y \neq x$. Thus y is not adjacent to x, which is a contradiction. Hence the result holds.

(iii) $\Rightarrow$ *(i)* Suppose that, for each vertex b in $\Gamma'(S)$, we have $|(b)| = 1$. Assume on the contrary that $M = S - T$ for some maximal ideal $M \in A$ and $|T| > 1$. So, for each x and y in T, we have $(x) = (y)$. This implies that $x = y$, which is a contradiction. □

Corollary 2.5. *If, for each maximal ideal $M \in A$, we have $M = S - \{a\}$, where $a^2 \in M$, then the graph $\Gamma'(S)\backslash M^*$ is complete .*

The following lemma, which is from [10], is needed in the rest of the paper.

Lemma 2.6. *Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$. Then the following statements hold:*

(i) *The cofactors of any two elements of $L(G)$ are equal.*
(ii) *The cofactor of any element of $L(G)$ equals the number of spanning trees of G.*
(iii) *For $n \geq 2$, we have $\kappa(K_n) = n^{n-2}$.*

In the corollary below, we investigate some basic properties of $\Gamma'(S)\backslash M^*$.

Corollary 2.7. *Let S be a semigroup with the identity element 1. Also, suppose that, for each maximal ideal $M \in A$, $M = S - \{a\}$, where $a^2 \in M$. Then the following statements hold:*

(i) $r(\Gamma'(S)\backslash M^*) = 1$.
(ii) $\chi(\Gamma'(S)\backslash M^*) = t$, *where* $|\Gamma'(S)\backslash M^*| = t$.
(iii) *The graph $\Gamma'(S)\backslash M^*$ is split.*
(iv) $\gamma(\Gamma'(S)\backslash M^*) = \lceil \frac{1}{12}(t-3)(t-4) \rceil$, *where* $|\Gamma'(S)\backslash M^*| = t \geq 3$.
(v) $\kappa(\Gamma'(S)\backslash M^*) = t^{t-2}$, *where* $|\Gamma'(S)\backslash M^*| = t \geq 2$.

Example 2.8. (i) *The graph $\Gamma'(\mathbb{Z}_n)$ is complete if and only if $n = 4$ or $n = p$, where p is a prime number.*
(ii) *If $n = p^t$, where p is prime and $t \geq 2$, then the graph $\overline{\Gamma'(\mathbb{Z}_n)}$ is complete.*

In the theorem below, we study the case that the graph $\Gamma'(S)\backslash M^*$ is a complete bipartite graph.

Theorem 2.9. *Let S be a semigroup with the identity element 1 and $A = \{M_\alpha, M_\beta\}$. Then $\Gamma'(S)\backslash M^*$ with at least three vertices is a complete bipartite graph with parts $M_\alpha\backslash M^*$ and $M_\beta\backslash M^*$ if and only if $M_\alpha = S - T$ and $M_\beta = S - T'$, where $|T| \geq 1$ and $|T'| > 1$.*

Proof. Suppose that $\Gamma'(S)\backslash M^*$ is a complete bipartite graph with parts $M_\alpha\backslash M^*$ and $M_\beta\backslash M^*$. Assume on the contrary that $M_\alpha = S - \{a_\alpha\}$ and $M_\beta = S - \{a_\beta\}$, where $a_\alpha^2 \in M_\alpha$ and $a_\beta^2 \in M_\beta$. Then, by Corollary 2.5, $\Gamma'(S)\backslash M^*$ is a complete graph, which is a contradiction.

Conversely, let $M_\alpha = S - T$, where $|T| \geq 1$ and $M_\beta = S - |T'|$, where $|T'| > 1$. Also, let x and y be two arbitrary distinct vertices in $M_\alpha\backslash M^*$. If x is adjacent to y in $M_\alpha\backslash M^*$, then $x \notin (y)$ and $y \notin (x)$. Now, since $x, y \in M_\alpha\backslash M^*$ and $M_\alpha \cup M_\beta = S$, we have x and y do not belong to M_β. So $x, y \in T'$. Therefore we have $(x) = (y)$, which is a contradiction. Similarly, no two distinct vertices in $M_\beta\backslash M^*$ are adjacent. Now, if $x \in M_\alpha\backslash M_\beta$ and $y \in M_\beta\backslash M_\alpha$, then clearly x and y are adjacent. Hence $\Gamma'(S)\backslash M^*$ is a complete bipartite graph with parts $M_\alpha\backslash M_\beta$ and $M_\beta\backslash M_\alpha$. This completes the proof. □

Corollary 2.10. *Let S be a semigroup with the identity element 1 and $A = \{M_\alpha, M_\beta\}$ such that $M_\alpha = S - \{a_\alpha\}$, where $a_\alpha^2 \in M_\alpha$ and $M_\beta = S - T$, where $|T| > 1$. If $|M_\beta\backslash M^*| = 1$, then $\Gamma'(S)\backslash M^*$ is a star graph.*

In the following corollary, we investigate some basic properties of $\Gamma'(S)$ in a special case.

Corollary 2.11. *Let S be a semigroup with the identity element 1 and $A = \{M_\alpha, M_\beta\}$ such that $M_\alpha = S - T$, where $|T| \geq 1$ and $M_\beta = S - T'$, where $|T'| > 1$. If $|M_\alpha \backslash M^*| = t_1$ and $|M_\beta \backslash M^*| = t_2$, then the following statements hold:*

(i) $r(\Gamma'(S)\backslash M^*) = 2$.
(ii) $\chi(\Gamma'(S)\backslash M^*) = 2$.
(iii) *If $t_1 < t_2$, then $\lambda(\Gamma'(S)\backslash M^*) = t_1$ and $\alpha(\Gamma'(S)\backslash M^*) = t_2$.*
(iv) $\gamma(\Gamma'(S)\backslash M^*) = \lceil \frac{1}{4}(t_1 - 2)(t_2 - 2) \rceil$, *where $t_1, t_2 \geq 2$.*
(v) *The eigenvalues of $\Gamma'(S)\backslash M^*$ are $\sqrt{t_1 t_2}$, $-\sqrt{t_1 t_2}$, and 0 with multiplicity $t_1 + t_2 - 2$ and $\varepsilon(\Gamma'(S)\backslash M^*) = 2\sqrt{t_1 t_2}$.*

The following lemma follows easily from Corollary 2.10 and Lemma 2.6.

Lemma 2.12. *Let S be a semigroup with identity element 1 and $A = \{M_\alpha, M_\beta\}$ such that $M_\alpha = S - \{a_\alpha\}$, where $a_\alpha^2 \in M_\alpha$ and $M_\beta = S - \overline{T}$ with $|\overline{T}| > 1$. If $|\overline{M}_\beta \backslash \overline{M}^*| = 1$, then $\kappa(\Gamma'(S)\backslash M^*) = 1$.*

Lemma 2.13. *If the graph $\Gamma'(S)\backslash M^*$ is n-partite, for some positive integer n, then $|A| \leq n$.*

Proof. Assume on the contrary that $|A| > n$. Since $\Gamma'(S)\backslash M^*$ is n-partite and $|A| > n$, there are M_α and M_β in A such that $M_\alpha = S - T$ and $M_\beta = S - T'$, where $|T|, |T'| \geq 1$ and there exist elements $a \in T$ and $b \in T'$ that belong to the same part. On the other hand, we have that a is adjacent to b, which is a contradiction. □

Now, in the following theorem, we show that $\mathrm{gr}(\Gamma'(S))$ is three, whenever S has more than two maximal ideals.

Theorem 2.14. *Let $A = \{M_\alpha \mid \alpha \in \Lambda\}$ be the set of all maximal ideals and $|A| \geq 3$. Then $\mathrm{gr}(\Gamma'(S)) = 3$.*

Proof. Suppose that $M_\alpha = S - T_1$, $M_\beta = S - T_2$ and $M_\gamma = S - T_3$ are distinct maximal ideals of S, where $|T_1| \geq 1$, $|T_2| \geq 1$ and $|T_3| \geq 1$. Consider $a \in T_1$, $b \in T_2$ and $c \in T_3$. Since $a \in T_1$, $M_\alpha \cup M_\beta = S$ and $M_\alpha \cup M_\gamma = S$, $a \in M_\beta \cap M_\gamma$. Similarly $b \in M_\alpha \cap M_\gamma$ and $c \in M_\alpha \cap M_\beta$. One can easily see that a is adjacent to both vertices b and c. We claim that the vertices b and c are adjacent. Suppose on the contrary that, b is not adjacent to c. Then $b \in (c)$ or $c \in (b)$. Without loss of generality, we may assume that $b = rc$ for some $r \in S$. Since $c \in M_\alpha \cap M_\beta$, we have $b \in M_\alpha \cap M_\beta$. So $b \in M_\beta$, which is impossible. Therefore we have the path $a - b - c - a$. Hence the result holds. □

Theorem 2.15. *Let S be a semigroup with the identity element 1. Then the following conditions are equivalent:*

(i) *$\Gamma'(S)\backslash M^*$ is complete bipartite.*
(ii) *$\Gamma'(S)\backslash M^*$ is bipartite*
(iii) *$\Gamma'(S)\backslash M^*$ contains no triangle.*

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.

$(ii) \Rightarrow (i)$ Suppose that $\Gamma'(S)\backslash M^*$ is a bipartite graph. Then, by Lemma 2.13, we have that $A = \{M_\alpha, M_\beta\}$ such that $M_\alpha = S - T$ and $M_\beta = S - T'$, where $|T|, |T'| \geq 1$. Now we put $V_1 = M_\alpha \backslash M_\beta$ and $V_2 = M_\beta \backslash M_\alpha$. Clearly every vertex in V_1 is adjacent to all vertices in V_2 and also every vertex in V_2 is adjacent to all vertices in V_1. Hence $\Gamma'(S)\backslash M^*$ is a complete bipartite graph.

$(iii) \Rightarrow (ii)$ Suppose that $\Gamma'(S)\backslash M^*$ contains no triangle. Then, by Theorem 2.14, S has exactly two maximal ideals, say M_α and M_β such that $M_\alpha = S - T$ and $M_\beta = S - T'$, where $|T|, |T'| \geq 1$. Now, if there exist two arbitrary distinct vertices x and y in $M_\alpha \backslash M^*$ such that x is adjacent to y, then $x \notin (y)$ and $y \notin (x)$. Since $x, y \in M_\alpha \backslash M^*$ and $M_\alpha \cup M_\beta = S$, we have that x and y do not belong to M_β. So $x, y \in T'$. Therefore $(x) = (y)$, which is a contradiction. □

Corollary 2.16. *Let S be a semigroup with the identity element 1, $S = S^2$ and every prime ideal of S is a maximal ideal. Then the following conditions are equivalent:*

(i) *$\Gamma'(S)\backslash Q^*$ is complete bipartite.*
(ii) *$\Gamma'(S)\backslash Q^*$ is bipartite*
(iii) *$\Gamma'(S)\backslash Q^*$ contains no triangle.*

In the next theorem, we study the girth of the graph $\Gamma'(S)\backslash M^*$.

Theorem 2.17. *In the cozero divisor graph of S, we have $gr(\Gamma'(S)\backslash M^*) = \infty$ or $\mathrm{gr}(\Gamma'(S)\backslash M^*) \leq 4$.*

Proof. Assume that $\mathrm{gr}(\Gamma'(S)\backslash M^*)$ is finite and also $\Gamma'(S)\backslash M^*$ doesn't contain a triangle. Then, in view of Theorem 2.15, $\Gamma'(S)\backslash M^*$ is bipartite. Let $x_0 - x_1 - \cdots - x_n - x_0$ be a cycle in $\Gamma'(S)\backslash M^*$ with $n > 5$. Also suppose that for $0 \leq i \leq n$, we can not eliminate x_i to obtain a shorter cycle. Now, since x_1 and x_{n-2} are not adjacent, by Theorem 2.2, there exists $a \in \Gamma'(S)\backslash M^*$ such that we have the path $x_1 - a - x_{n-2}$. So we have the cycle $x_1 - x_2 - \cdots - x_{n-2} - a - x_1$, which is impossible. Therefore $\mathrm{gr}(\Gamma'(S)\backslash M^*) \leq 5$. Since $\Gamma'(S)\backslash M^*$ is bipartite, we have $\mathrm{gr}(\Gamma'(S)\backslash M^*) = 4$. Hence the result holds. □

In the following theorem, we investigate some cases that $\Gamma'(S)\backslash M^*$ is not Eulerian.

Theorem 2.18. *If S contains a principal maximal ideal M such that $|W(S)\backslash M|$ is an odd number, then $\Gamma'(S)\backslash M^*$ is not Eulerian.*

Proof. Suppose that $M = (x)$ is a principal maximal ideal of S. Then, for all $y \in M\backslash\{x\}$, we have x and y are not adjacent. Also, for all $z \in W(S)\backslash M$, x is adjacent to z. Therefore the degree of x is equal to $|W(S)\backslash M|$, which is an odd number. Hence, by [13, Theorem 4.1], $\Gamma'(S)\backslash M^*$ is not Eulerian. □

Example 2.19. *Let p and q be distinct prime numbers. Then we have the following statements.*

(i) *If $n = p^t$, where $t \geq 2$, then $\Gamma'(\mathbb{Z}_n)$ ia an empty graph, which is not Eulerian.*
(ii) *If $n = 2p$, where $p \neq 2$, then $\Gamma'(\mathbb{Z}_n)$ is a star graph with center p. Hence $\Gamma'(\mathbb{Z}_n)$ is not Eulerian.*

(iii) *If $n = pq$, where $p, q > 2$, then $\Gamma'(\mathbb{Z}_n)$ is a complete bipartite graph with parts $A = \{x \in \Gamma'(\mathbb{Z}_{pq}) : p|x\}$ and $B = \{y \in \Gamma'(\mathbb{Z}_{pq}) : q|y\}$. Hence $\Gamma'(\mathbb{Z}_n)$ is isomorphic to $K_{p-1,q-1}$, which is an Eulerian graph.*

Proof. (i) It is clear.

(ii) We claim that the vertex p is adjacent to all vertices y in $\Gamma'(\mathbb{Z}_{2p})$, where $y \neq p$. Since $(y, 2p) \neq 1$, one can easily see that $2 \mid y$. Thus $y = 2z$ for some $z \in \mathbb{Z}_{2p}$. Therefore we have that $p \notin (y)$ and $y \notin (p)$, which implies that p and y are adjacent. Now, suppose that $2k_1$ and $2k_2$ are arbitrary distinct vertices in $\Gamma'(\mathbb{Z}_{2p})$. Clearly $k_1, k_2 < p$. Assume that $m = k_1 p + k_2 t$, where t is the inverse of k_1 in the field $\mathbb{Z}_p$. Then we have $k_2 \equiv k_1 m \pmod{p}$, and so $2k_2 \equiv 2k_1 m \pmod{2p}$. This means that $2k_2 \in (2k_1)$, and so the vertices $2k_1$ and $2k_2$ are not adjacent. Hence $\Gamma'(\mathbb{Z}_{2p})$ is a star graph with center p.

(iii) Put $A = \{x \in \mathbb{Z}_{pq} : p \mid x\}$ and $B = \{y \in \mathbb{Z}_{pq} : q \mid y\}$. Then clearly $|A| = q - 1$ and $|B| = p - 1$. Also $A \cup B = V(\Gamma'(\mathbb{Z}_{pq}))$ and $A \cap B = \emptyset$. If $x, x' \in A$, then $x = pt$ and $x' = pt'$, and so $x \in (x')$ or $x' \in (x)$. Hence x is not adjacent to x' in A. Similarly no two distinct vertices in B are adjacent. Now we claim that each vertex in A is adjacency to all vertices in B. Let $x \in A$ be an arbitrary vertex in $\Gamma'(\mathbb{Z}_n)$. We show that x is adjacent to y for every $y \in B$. Since $x \in A$, we have $x = pz$ for some $z \in \mathbb{Z}_q$. Also $y \in B$ implies that $y = qz'$ for some $z' \in \mathbb{Z}_p$. Thus $x \notin (y)$ and $y \notin (x)$. Hence the result holds. □

Theorem 2.20. *Let S be a finite semigroup with two maximal ideals M_α and M_β such that $|M_\alpha| = |M_\beta|$. Then $\Gamma'(S)\backslash M^*$ is Hamiltonian.*

Proof. Suppose that $M_\alpha = S - T$ and $M_\beta = S - T'$ are two maximal ideals, where $|T| = |T'| \geq 1$. Then since $M_\alpha \cup M_\beta = S$, we set $M_\alpha \backslash M^* := \{a_1, \ldots, a_n\}$ and $M_\beta \backslash M^* := \{b_1, \ldots, b_n\}$. Then one can easily see that

$$a_1 - b_1 - a_2 - \cdots - a_n - b_n - a_1$$

is a Hamiltonian cycle in $\Gamma'(S)\backslash M^*$. Hence the result holds. □

Example 2.21. *Let p and q be two distinct prime numbers. Then the graph $\Gamma'(\mathbb{Z}_n)$ is not Hamiltonian if $n = p^t$, where $t \geq 2$ or $n = 2p$, where $p \neq 2$ or $n = pq$, where $p, q > 2$.*

Now, in the following theorem, we study the planarity of $\Gamma'(S)$.

Theorem 2.22. *Let $A = \{M_\alpha \mid \alpha \in \Lambda\}$ be the set of all maximal ideals of S and $|A| \geq 5$. Then $\Gamma'(S)$ is not planar.*

Proof. Suppose that $M_1, \ldots, M_5$ are distinct maximal ideals of S such that $M_i = S \setminus T_i$, for $i = 1, \ldots, 5$, and $|T_i| \geq 1$. Now, we choose, $a_1 \in T_1$, $a_2 \in T_2$, $a_3 \in T_3$, $a_4 \in T_4$ and $a_5 \in T_5$. Since $a_1 \in T_1$ and $M_1 \cup M_i = S$ for $2 \leq i \leq 5$, we have $a_1 \in \bigcap_{i=2}^{5} M_i$. Similarly $a_j \in \bigcap_{i=1,\ i\neq j}^{5} M_i$, for $j = 2, \cdots, 5$. Then one can easily see that the vertex set $\{a_1, a_2, a_3, a_4, a_5\}$ forms a complete subgraph of $\Gamma'(S)$, which is isomorphic to K_5. Hence, by Kuratowaski's Theorem, $\Gamma'(S)$ is not planar. □

In the following theorems, we study the chromatic, clique and independence numbers of the graph $\Gamma'(S)\backslash M^*$.

Theorem 2.23. *Let S be a semigroup with the identity element* 1. *Then* $\chi(\Gamma'(S)\backslash M^*) = 2$ *if and only if* $\omega(\Gamma'(S)\backslash M^*) = 2$.

Proof. If $\chi(\Gamma'(S)\backslash M^*) = 2$, then since $|A| > 1$ and $\omega(\Gamma'(S)\backslash M^*) \leq \chi(\Gamma'(S)\backslash M^*)$, we have $\omega(\Gamma'(S)\backslash M^*) = 2$.

Conversely, suppose that $\omega(\Gamma'(S)\backslash M^*) = 2$. Then we have $|A| = 2$. Now, if $\chi(\Gamma'(S)\backslash M^*) > 2$, then $\Gamma'(S)\backslash M^*$ is not bipartite. So, by Theorem 2.15, it contains a triangle. This implies that $\omega(\Gamma'(S)\backslash M^*) \geq 3$, which is impossible. Hence the result holds. □

Theorem 2.24. *Let S be a semigroup with the identity element* 1 *and $M_\alpha = S - T_\alpha$ for some sets T_α. Then* $\alpha(\Gamma'(S)\backslash M^*) \geq \max\{|T_\alpha| \mid \alpha \in \Lambda\}$.

Proof. Clearly, for each $\alpha \in \Lambda$, $T_\alpha \subseteq W(S)\backslash M^*$. Also, for each two distinct vertices x and y in T_α, we have $(x) = (y)$. This implies that x and y are not adjacent in $\Gamma'(S)\backslash M^*$. Therefore T_α forms an independent set. Thus we have

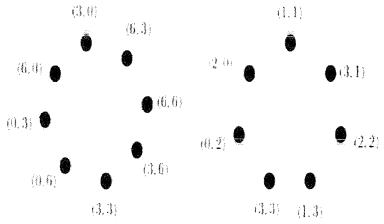
$$\alpha(\Gamma'(S)\backslash M^*) \geq \max\{|T_\alpha| \mid \alpha \in \Lambda\}.$$

□

In [1] and [2], Emad Abu Osba, Salah Al-Addasi and Nafez Abu Jaradeh introduced and investigated some properties of the zero divisor graph for the ring of Gaussian integers module n, where n is a positive integer. Recall that the semigroup of Gaussian integers module n, denoted by $\mathbb{Z}_n[i]$, is the set $\mathbb{Z}_n[i] = \{a + ib \ : \ a, b \in \mathbb{Z}_n\}$, with the complex multiplicative operation. We denote the element $a+ib$ in $\mathbb{Z}_n[i]$, by the ordered pair (a, b). In the next example, we study the cozero divisor graph of the semigroup $\mathbb{Z}_n[i]$, in the case that $n = p^t$, where p is a prime number. The cozero divisor graph $\Gamma'(\mathbb{Z}_n[i])$ is an undirected graph whose vertex set contains all non-zero and non-unit elements of $\mathbb{Z}_n[i]$, and two distinct vertices (a, b) and (c, d) are adjacent if and only if $(a, b) \notin < (c, d) >$ and $(c, d) \notin < (a, b) >$, where, for $(e, f) \in \mathbb{Z}_n[i]$, $< (e, f) >$ is the ideal $(e, f)\mathbb{Z}_n[i]$. By [1], we have $V(\Gamma'(\mathbb{Z}_n[i])) = \{(a, b) \ : \ a, b \in \mathbb{Z}_n, \ a^2 + b^2 \equiv \text{zero divisor of } \mathbb{Z}_n\}$. Clearly $V(\Gamma'(\mathbb{Z}_2[i])) = \{(1, 1)\}$, and so $\Gamma'(\mathbb{Z}_2[i])$ is isomorphic to K_1. Also, by [1], if $q \equiv 3 \pmod 4$, then $\mathbb{Z}_q[i]$ is an integral domain. Therefore $\Gamma'(\mathbb{Z}_q[i])$ is a null graph.

Example 2.25. *If $n = p^t$, where $t \geq 2$ and p is prime, then $\Gamma'(\mathbb{Z}_n[i])$ is an empty graph.*

Proof. Suppose that (x, y) and (z, w) are arbitrary distinct vertices in $\Gamma'(\mathbb{Z}_n[i])$. Then $x^2 + y^2 \equiv$ zero divisor of $\mathbb{Z}_{p^t} \pmod{p^t}$ and $z^2 + w^2 \equiv$ zero divisor of $\mathbb{Z}_{p^t} \pmod{p^t}$. Hence we have $x^2 + y^2 \equiv$ zero divisor of $\mathbb{Z}_p \pmod p$ and $z^2 + w^2 \equiv$ zero divisor of $\mathbb{Z}_p \pmod p$. Thus there exists an element $(r, s) \in \mathbb{Z}_{p^t}[i]$ such that $(x, y) = (r, s)(z, w)$ or $(z, w) = (r, s)(x, y)$. Therefore (x, y) and (z, w) are not adjacent in $\Gamma'(\mathbb{Z}_{p^t}[i])$. Hence the result holds. □



$\Gamma'(\mathbb{Z}_9[i])$ $\Gamma'(\mathbb{Z}_4[i])$

Figure 1

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GENERALIZED FRATTINI IDEALS OF n-LIE ALGEBRAS

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Abstract

As an analogue of generalized Frattini ideals of a Lie algebra, the generalized Frattini ideals of an n-Lie algebra are investigated. Precisely, we give elementary properties of generalized Frattini ideals of n-Lie algebras.

Key words: n-Lie algebras; generalized Frattini ideals
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§1 Introduction

In recent years, much interest has been shown in the study of n-Lie algebras, which is due to various speculations concerning dynamics, geometry, and mathematical curiosity as well. In 1985, Filippov introduced the concept of n-Lie algebras [7]. An n-ary multi-linear and skew symmetric operation $[x_1, ..., x_n]$ satisfying the following generalized Jacobi identity was considered:

$$[[x_1, ..., x_n], y_2, ..., y_n] = \sum_{i=1}^{n} [x_1, ...[x_i, y_2, ..., y_n], ..., x_n].$$

and the first basic example of n-Lie algebra was given. In this paper we always assume that A is a finite-dimensional n-Lie algebra over an algebraically closed field $\mathbb{F}$ with characteristic zero. An n-Lie algebra is a vector space A over a field $\mathbb{F}$ with an n-ary multi-linear operation $[, ...,]$ satisfying:

$$[x_1, ..., x_n] = (-1)^{\tau(\sigma)} [x_{\sigma(1)}, ..., x_{\sigma(n)}],$$

$$[[x_1, ..., x_n], y_2, ..., y_n] = \sum_{i=1}^{n} [x_1, ...[x_i, y_2, ..., y_n], ..., x_n],$$

where runs over the symmetric group S_n and the number $\tau(\sigma) \in \{0, 1\}$ depending on the parity of the permutation σ.

A major subject of study in algebra is the intersection of all maximal "things". In groups, Giovanni Frattini investigated the intersection of all maximal subgroups, now called the Frattini subgroup. The Frattini subgroup, denoted Frat(G), for a group G has been continually investigated since Frattini published his results in a paper in 1885. In the first half of the 20^{th} century, various authors investigated this concept in group theory. The theory of Frattini subgroup and generalized Frattini subgroups of a

group is well developed and has proved useful in the study of certain types of problems in group theory (see, e.g., [3, 4, 8]). The corresponding concept for algebras has widely recognized and is of independent interest(see, e.g., [1, 2, 5-6, 9-15]).

The purpose of the present paper is to give some generalized Frattini ideals for n-Lie algebras. As an analogue of generalized Frattini ideals of a Lie algebra, the generalized Frattini ideals of an n-Lie algebra are investigated. The fundamental facts about n-Lie algebras are collected in section 2. We then in section 3 establish some elementary results on generalized Frattini ideals of n-Lie algebras.

§2 Preliminaries

To begin, all n-Lie algebras that are investigated in this paper will be finite-dimensional. We will first introduce definitions and notations that will be used throughout this paper. Using the same notation as David Towers in [13], the Frattini subalgebra, $F(L)$, is the intersection of all maximal subalgebras in an n-Lie algebra L. As this is not always an ideal, we refer to the Frattini ideal, $\phi(L)$, as the largest ideal of L contained in $F(L)$.

Definition 2.1. Let L be an n-Lie algebra. A subspace B of L is called a subalgebra if

$$[B, B, ..., B] \subseteq B.$$

A subspace I is called an ideal of n-Lie algebras if

$$[I, L, ..., L] \subseteq I.$$

If I satisfies

$$[I, I, L, ..., L] = 0,$$

then I is called an abelian ideal of L.

Definition 2.2. For a subalgebra H of L, denote

$$N_L(H) = \{x \in L, [x, H, ..., H] \subseteq H\},$$

$$Z_L(H) = \{x \in L, [x, H, L, ..., L] = 0\};$$

it is clear that if H is an ideal, then $N_L(H)$ and $Z_L(H)$ are ideals.

Definition 2.3. An n-Lie algebra L is said to be decomposable if there exist nonzero ideals L_1, L_2 such that

$$L = L_1 \oplus L_2,$$

and $[L_1, L_2, L, ..., L] = 0$. Otherwise, we say that L is indecomposable. Clearly, if L is a simple n-Lie algebra, then L is indecomposable.

Definition 2.4. An ideal I is called nilpotent if $I^r = 0$ for some $r \geq 0$, where $I^0 = I$ and define

$$I^{s+1} = [I^s, I, L, ..., L],$$

for some $s \geq 0$. When $L = I$, then L is called nilpotent.

An ideal I is called solvable if $I^{(r)} = 0$ for some $r \geq 0$, where $I^{(0)} = I$ and define

$$I^{(s+1)} = [I^{(s)}, I^{(s)}, L, ..., L],$$

for some $s \geq 1$. When $L = I$, then L is called solvable.

Definition 2.5. A subalgebra C of an n-Lie algebra L is called a Cartan subalgebra of L if C is nilpotent and $N_L(C) = C$. Following this definition, a proper ideal H is a generalized Frattini ideal of L if for each ideal K in L and each Cartan subalgebra C of K, whenever $L = H + N_L(C)$, then $L = N_L(C)$.

Definition 2.6. The ideal I of n-Lie algebra L is called the Jacobson radical, if I is the intersection of all maximal ideals of L, denoted by $J(A)$.

Definition 2.7. If K is an ideal in L, then $K^w = \bigcap_{i=1}^{\infty} K^i$.

We define the lower central series of L inductively by letting $L_1 = L$ and $L_i = [L, ..., L, L_{i-1}, L_{i-1}]$ for all $i > 1$. Similarly, we define the upper central series of L by letting $Z_0(L) = 0$ and $Z_i(L)$ be the ideal of L such that $Z(L/Z_{i-1}(L)) = Z_i(L)/Z_{i-1}(L)$ for $i \geq 1$. If A and B are ideals of L such that $L = A + B$ and $A \cap B = 0$, we write $L = A \oplus B$.

§3 Elementary results on generalized Frattini ideals

Theorem 3.1. Let H be a generalized Frattini ideal of an n-Lie algebra L. Then

(1) H is nilpotent.

(2) An ideal of L contained in H is a generalized Frattini ideal.

(3) $H + \phi(L)$ a is generalized Frattini ideal.

(4) $H + Z(L)$ is a generalized Frattini ideal, whenever it is a proper ideal.

Proof. (1) Let C be a Cartan subalgebra of H and H be a generalized Frattini ideal, then $L = H + N_L(C)$. Since H is a generalized Frattini ideal, $L = N_L(C)$. Hence $N_H(C) = H$. As C is a Cartan subalgebra of H, $N_H(C) = C$. Thus $H = C$. Since C is nilpotent, H is nilpotent.

(2) Let N be an ideal of L such that N is contained in H. Let K be an ideal of L with C, Cartan subalgebra of K, such that $L = N + N_L(C)$. But $L = H + N_L(C)$, hence $L = N_L(C)$. So N is a generalized Frattini ideal.

(3) Let $\phi(L)$ be the Frattini ideal. Suppose that K is an ideal of L with Cartan subalgebra C such that $H + \phi(L) + N_L(C) = L$. Let M is a maximal ideal with $H + N_L(C) \subseteq M$. But $\phi(L) \subseteq M$ which contradicts $H + \phi(L) + N_L(C) = L$. So no such M exists and $L = H + N_L(C)$. Since H is a generalized Frattini ideal, this implies $L = N_L(C)$. Thus $H + \phi(L)$ is a generalized Frattini ideal.

(4) Consider $H + Z(L) + N_L(C) = L$. Since $Z(L)$ is contained in every normalizer, $Z(L) \subseteq N_L(C)$. So $H + Z(L) + N_L(C) = H + N_L(C) = L$. Since H is a generalized Frattini ideal, $N_L(C) = L$. Thus $H + Z(L)$ is a generalized Frattini ideal.

Corollary 3.2. The Frattini ideal $\phi(L)$ and the center $Z(L)$ are generalized Frattini ideals of L.

Proof. The proof is the same as in Theorem 3.1 parts (3) and (4) and take $H = 0$.

Theorem 3.3. Let H be a generalized Frattini ideal of L. If K is an ideal of L and K/H is nilpotent, then K is nilpotent.

Proof. Let C be Cartan in K with $H \subseteq K$ such that $N_K(C) = C$. For any $k + H \in N_{K/H}(C + H/H)$, we have $[k + H, c + H, ..., c + H] = [k, c, ..., c] + H \in (C+H)/H$, where $c \in C$. Thus, $[k, c, ..., c] \in C$. Therefore, $k \in N_K(C) = C$. Then we have $N_{K/H}(C + H/H) \subseteq C + H/H$. Since $[c + H, c + H, ..., c + H] = [c, c, ..., c] + H \in (C + H)/H$ for each $c + H \in C + H/H$, $C + H/H \subseteq N_{K/H}(C + H/H)$. Then we have $C + H/H$ is Cartan in K/H. By definition, $N_{K/H}(C + H/H) = C + H/H$. Since K/H is nilpotent, $K/H \subseteq N_{K/H}(C + H/H)$ and $N_{K/H}(C + H/H) \neq C + H/H$ unless $K/H = C + H/H$. So $K/H = C + H/H$. Hence $K = C + H$. Since $L = N_L(C) + K = N_L(C) + C + H = N_L(C) + H$ and H is a generalized Frattini ideal, $L = N_L(C)$ and $K = N_K(C) = C$ is nilpotent.

Corollary 3.4. Let H be a generalized Frattini ideal of L. Then L is nilpotent if and only if L/H is nilpotent.

Proof. Let k be the positive integer such that $L^k = 0$. Then

$$(L/H)^{k+1} = [(L/H)^k, L/H, ..., L/H] = [L^k, L, ..., L]/H = 0.$$

Thus, L/H is nilpotent. Since L is an ideal and by the theorem 3.3, we have L is nilpotent.

Corollary 3.5. An n-Lie algebra L is nilpotent if and only if L^1 is a generalized Frattini ideal.

Proof. Suppose L^1 is a generalized Frattini ideal. Then L/L^1 is nilpotent. By Theorem 3.3, L is nilpotent. Conversely, suppose that L is nilpotent. Then for any ideal H of L and $L^1 \subseteq H$, H/L^1 nilpotent implies that H is nilpotent. Thus L^1 is a generalized Frattini ideal.

Theorem 3.6. Let H be a generalized Frattini ideal of L. If K is an ideal in L such that $K^w \subseteq H$, then $K^w = 0$ and K is nilpotent.

Proof. Since $K^w \subseteq H$, K/H is nilpotent. Thus K is nilpotent by Theorem 3.3 and hence $K^w = 0$.

Theorem 3.7. Let H be an ideal of L. H is a generalized Frattini ideal of L if and only if for all ideals J of L such that $H \subseteq J$ and J/H is nilpotent, then J is nilpotent.

Proof. Let H be a generalized Frattini ideal of L. By Theorem 3.3, if $H \subseteq J$ and J/H is nilpotent, then J is nilpotent and the condition holds. Conversely, suppose the condition holds. Let K be an ideal of L and C a Cartan subalgebra of K such that $L = N_L(C) + H$, $H + C/H$ is Cartan in $K + H/H$, $N_{K/H}(C + H/H) = C + H/H$. Since $[N_L(C) + H, C + H, ..., C + H] \subseteq C + H$, $L = N_L(C) + H$. Then $C + H/H$ is an ideal in L/H. As $[C + H/H, K + H/H, ..., K + H/H] \subseteq [C + H/H, L/H, ..., L/H] \subseteq C + H/H$, $H + C/H$ is an ideal in $K + H/H$. But $N_{K/H}(C + H/H) = C + H/H = K + H/H$. Therefore, $K + H/H$ is nilpotent, and so $K + H$ is nilpotent. Thus K is nilpotent and $C = K$. Since $[L, K, .., K] \subseteq [L, L, ..., K] \subseteq K$, $L \subseteq N_L(K)$. Hence $N_L(C) = N_L(K) = L$. Therefore, H is a generalized Frattini ideal.

Corollary 3.8. A proper ideal K of L is nilpotent if and only if K^1 is a generalized Frattini ideal of L.

Proof. If K^1 is a generalized Frattini ideal, then K/K^1 is nilpotent. Thus

K is nilpotent. Conversely, suppose K is nilpotent. Then $K^1 = \phi(K) \subseteq \phi(L)$. If J is an ideal of L such that J/K^1 is nilpotent, then $J/\phi(K)$ is nilpotent. Let k be the positive integer such that

$$(J/\phi(K))^{k+1} = [(J/\phi(K))^k, J/\phi(K), L/\phi(K), ..., L/\phi(K)] = 0.$$

Thus $[J^k, J, L, ..., L] \subseteq \phi(k) \subseteq \phi(L)$. So then $(J + \phi(L)/\phi(L))^{k+1} = 0$. Therefore $J + \phi(L)/\phi(L)$ is nilpotent and J is nilpotent. So by theorem 3.7, K^1 is a generalized Frattini ideal.

Consequently, we get the following.

Corollary 3.9. All proper ideals of nilpotent n-Lie algebras are generalized Frattini ideals.

Theorem 3.10. Let H be a generalized Frattini ideal of L and let K be an ideal of L containing H. Then K/H is a generalized Frattini ideal of L/H if and only if K is a generalized Frattini ideal of L.

Proof. Suppose that K is a generalized Frattini ideal of L. Let J/H be an ideal of L/H such that $\frac{J/H}{K/H}$ is nilpotent. Since $\frac{J/H}{K/H} \cong J/K$, J/K is nilpotent. Since K is a generalized Frattini ideal, J is nilpotent. But then J/H is nilpotent. Thus by Theorem 3.7, K/H is a generalized Frattini ideal of L/H. Conversely, suppose that K/H is a generalized Frattini ideal of L/H. Let J be an ideal of L such that J/K is nilpotent. Since $\frac{J/H}{K/H} \cong J/K$, $\frac{J/H}{K/H}$ is nilpotent. So by Corollary 3.4, J/H is nilpotent. So then J is nilpotent since H is a generalized Frattini ideal. Again by Theorem 3.7, K is a generalized Frattini ideal of L. Recall that the nilpotent radical, $Nil(L)$, is the unique nilpotent ideal which contains all nilpotent ideals of L and that the radical, $Rad(L)$, is the maximal solvable ideal containing all solvable ideals of an n-Lie algebra L. The next theorems involve the nilpotent radical and the radical as well as provide a necessary and sufficient condition for the nilpotent radical to be a generalized Frattini ideal of L.

Theorem 3.11. If $Nil(L)$ is a generalized Frattini ideal, then every solvable ideal of L is nilpotent.

Proof. Let H be a solvable ideal of L and let k be the smallest positive integer such that $H^{(k+1)} = 0$, $[H^{(k)}, H^{(k)}, L, ..., L] = 0$. So $(H^{(k)})^2 = 0$. Thus $H^{(k)}$ is nilpotent and so contained in $Nil(L)$. Since $H^{(k)}$ is contained in $Nil(L)$, $H^{(k)}$ is a generalized Frattini ideal. Since $(H^{(k-1)}/H^{(k)})^2 = [H^{(k-1)}/H^{(k)}, H^{(k-1)}/H^{(k)}, L/H^{(k)}, ..., L/H^{(k)}] = [H^{(k-1)}, H^{(k-1)}, L, ..., L]/H^{(k)} = 0$, $H^{(k-1)}/H^{(k)}$ is nilpotent. By Theorem 3.3, $H^{(k-1)}$ is nilpotent. Hence $Nil(L)$ contains $H^{(k-1)}$. Continuing in this way we get $H^{(1)}$ is contained in $Nil(L)$. Thus $H^{(1)}$ is a generalized Frattini ideal. So then $H/H^{(1)}$ is nilpotent and $H^{(1)}$ is a generalized Frattini ideal, this implies H is nilpotent.

Corollary 3.12. If $Nil(L)$ is a generalized Frattini ideal of L, then L can not be solvable.

Proof. If L is solvable, then by Theorem 3.11, L is nilpotent and $L = Nil(L)$. Then $Nil(L)$ is not a proper ideal, so it can not be a generalized Frattini ideal.

Theorem 3.13. If H is a generalized Frattini ideal of L, then $Nil(L/H) = Nil(L)/H$.

Proof. Since H is a generalized Frattini ideal of L, H is nilpotent. So $H \subseteq Nil(L)$. Then $Nil(L)/H$ is nilpotent, so $Nil(L)/H \subseteq Nil(L/H)$. On the other hand, if $B/H = Nil(L/H)$, then B is nilpotent by Corollary 3.4. Then $B/H \subseteq Nil(L)/H$ and $B/H \subseteq Nil(L)/H$. Hence the result holds.

Theorem 3.14. Let L be non-nilpotent. If $Rad(L) = Nil(L)$, then $Nil(L)$ is a generalized Frattini ideal of L.

Proof. Let N be an ideal of L containing $Nil(L)$ such that $N/Nil(L)$ is nilpotent. Then $N/Nil(L)$ is solvable and N is solvable. Hence $N = Nil(L) = Rad(L)$. Then N is nilpotent. Therefore, $Nil(L)$ is a generalized Frattini ideal of L by Theorem 3.7.

Theorem 3.15. Let L be non-nilpotent. $Nil(L)$ is a generalized Frattini ideal of L if and only if $Nil(L) = Rad(L)$.

Proof. If $Nil(L) = Rad(L)$, then by Theorem 3.14, $Nil(L)$ is a generalized Frattini ideal. If $Nil(L)$ is a generalized Frattini ideal of L, then $Rad(L)$ is nilpotent by Theorem 3.11. Hence $Rad(L) = Nil(L)$.

Proposition 3.16. If L is a nilpotent n-Lie algebra and H is a proper ideal of L, then $H \cap Z(L) \neq 0$.

Proof. It is sufficient to show that any minimal ideal M of L is contained in $Z(L)$. There exists a non-negative integer i such that $M \cap Z_i(L) = 0$ and $M \cap Z_{i+1}(L) \neq 0$. Then $M \cap Z_{i+1}(L) = M$ since M is minimal. Then

$$[L, ..., L, M] \subseteq [L, ..., L, Z_{i+1}(L)] \subseteq Z_i(L).$$

But $[L, ..., L, M] \subseteq M$. Hence $[L, ..., L, M] \subseteq Z_i(L) \cap M = 0$ and $M \subseteq Z(L)$.

Proposition 3.17. If L is an n-Lie algebra and M is a minimal ideal of L, then $N(L) \subseteq Z_L(M)$.

Proof. If $M \cap N(L) = 0$, then $[M, ..., M, N(L)] = 0$ and $N(L) \subseteq Z_L(M)$. If $M \cap N(L) \neq 0$, then $M \subseteq N(L)$ since M is minimal. Then by Proposition 3.16, $M \cap Z(N(L)) \neq 0$. Also $M \cap Z(N(L))$ is an ideal of L because $Z(N(L))$ is a characteristic ideal of $N(L)$. Since M is minimal, $M \subseteq Z(N(L))$. This implies $N(L) \subseteq Z_L(M)$.

We define the socle of L, $Soc(L)$, to be the union of all minimal ideals. It is easily verified that $Soc(L)$ is the direct sum of minimal ideals. We also define the abelian socle of L, $Asoc(L)$, to be the union of all minimal abelian ideals. $Asoc(L)$ is the direct sum of minimal abelian ideals.

Proposition 3.18. Let L be an n-Lie algebra. Then $Asoc(L) \subseteq N(L) \subseteq Z_L(Soc(L))$. *Proof.* The first inclusion is obvious. To show the second, we write

$$Soc(L) = \bigoplus_{i=1}^{n} B_i.$$

where the B_i, $i = 1, ..., n$, are minimal ideals of L. By Proposition 3.17, $N(L) \subseteq Z_L(M)$ for $i = 1, ..., n$. Then

$$N(L) \subseteq \bigcap_{i=1}^{n} Z_L(B_i) = Z_L(\bigoplus_{i=1}^{n} B_i) = Z_L(Soc(L)).$$

Theorem 3.19. Let L be an n-Lie alegbra with $\phi(L) = 0$. Then $Z_L(Soc(L))$ $=Asoc(L) = N(L)$.

Proof. By Proposition 3.18, it is only necessary to show that $Z_L(Soc(L)) = Asoc(L)$. For convenience, let $A = Asoc(L)$ and $C = Z_L(Soc(L))$. Since A is abelian and $\phi(L) = 0$, there exists a subalgebra B of L such that $L = A \oplus B$. Then $A + (B \cap C) = (A + B) \cap (A + C) = L \cap C = C$. It is sufficient to show that $B_1 = B \cap C = 0$. Suppose $B_1 \neq 0$. We claim that B is an ideal in L. In order to see this, we note that

$$[L, L, ..., B_1] \subseteq [A + B, A + B, ..., B_1].$$

But $[A_i, A_i, ..., B_1] = 0$ for some $A_i = A$ because $A \subseteq Soc(L)$ and $B_1 \subseteq Z_L(Soc(L))$. Also, $[B, B, ..., B_1] \subseteq [B, B, ..., B] \subseteq B$ while $[B, B, ..., B_1] \subseteq [B, B, ..., C] \subseteq C$ because C is an ideal. Thus $[B, B, ..., B_1] \subseteq B \cap C = B$ and hence $[L, L, ..., B_1] \subseteq B_1$. Therefore B_1 is an ideal of L and must contain a minimal ideal I of L. If I is not abelian, then $[I, I, L, ..., L] = I$ and $I \nsubseteq Z_L(Soc(L))$. This contradicts $I \subseteq B_1 \subseteq Z_L(Soc(L))$. Therefore I must be abelian. However, a minimal abelian ideal must be in A. This again is a contradiction since $A \cap B_1 = 0$. Therefore $B_1 = 0$ and $Asoc(L) = Z_L(Soc(L))$.

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ISO-HILBERT ISO-SPACES*

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Abstract

In this article we make lift of Hilbert spaces to so - called iso-Hilbert iso-spaces. They are given main definitions and main conceptions for such iso-spaces and they are made comparisons with the real or complex Hilbert spaces.

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1 Introduction

Numerous aspects help the bradening of the scattering theory to incorporate non - Hamiltonian effects, effects which can not be represented using the conventional Hamiltonian. Following decades of research this incorporated required the construction by various authors of a new mathematics, well known as isomathematics and proposed by Santilli [1] in 1978, subsequently studied by the same author and numerious pure and applied mathematicians as: S. Okubo, H. Myung, M. Tomber, Gr. Tsagas, D. Sourlas, J. Kadeisvili, A. Aringazin, A. Kirhukin, R. Ohemke, G. Wene, G. M. Benkart, J. Osborn, D. Britten, J. Lohmus, E. Paal, L. Sorgsepp, D. Lin, J. Voujouklis, P. Broadbridge, P. Chernooff, J. Sniatycku, S. Guiasu, E. Prugovecki, A. Sagle, C. Jiang, R. Falcon Ganfornina, J. Nunez Valdes, A. Davvaz, and others.

As a result of these efforts, the new mathematics can be constructed via the systematic application of axiom - preservibg liftings, called isotopies, of the totality of the mathematics of quantum mechanics, inluding all its operators and all its operations, icluding the isotopic lifting of numbers, functinal analysis, differential calculas, geometries, topologies, Lie theory and others [2], [3], [4]. In [5] is shown that the isotopies can be very easly constructed using the application of nonunitary transforms to the totality of the formalism of the conventional scattering theory.

The physical needs for isomathematics have been indicated in [5], and consists in the necessity for a representation of non-Hamiltonian scattering effects in a form that is invariant over time so as to admit the same numerical predictions under the same conditions at different times. Following the study of all possible alternatives, the latter condition required the representation of non-Hamiltonian scattering effects with an axiom-preserving generalization of the trivial (positive-deffinite) unit of quantum mechanics $\hat{h} = 1$ into the most general possible (positive-deffinite as a condition to characterize an isotopy), integro-differential operator $\hat{I}$ which is as positive - definite as $+1$, functional depending of local variables, that is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1\hat{>}0 \longrightarrow \hat{I}(t, r, p, a, E, \ldots) = \frac{1}{\hat{T}}\hat{>}0$$

and it is called Santilli isounit. Santilli introduced a generalization called

lifting of the conventional associative product ab into the form

$$ab \longrightarrow a\hat{\times}b = a\hat{T}b$$

called isoproduct for which:

$$\hat{I}\hat{\times}a = \frac{1}{\hat{T}}\hat{T}a = a\hat{\times}\hat{I} = a\hat{T}\frac{1}{\hat{T}} = a.$$

for every element a of the field of real numbers, complex numbers and quaternions.

The Santilli isonumbers are defined as follows: for given real number or complex number or quaternion a,

$$\hat{a} = a\hat{I},$$

with isoproduct

$$\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b} = a\frac{1}{\hat{T}}\hat{T}b\frac{1}{\hat{T}} = ab\frac{1}{\hat{T}} = \hat{ab}.$$

If $a \neq 0$ the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$ or $\hat{I}\diagup\hat{a}$.

With $\hat{F}_{\mathbb{R}}$ we will denote the field of iso-reals. Below we will suppose that $\hat{T}_1 \in \hat{F}_{\mathbb{R}}$, $\hat{T}_1 > 0$ is an isotopic element which corresponds of $\hat{F}_{\mathbb{R}}$.

In this article we make lift of Hilbert spaces to so - called iso-Hilbert iso-spaces. They are given main definitions and main conceptions for such iso-spaces and they are made comparisons with the real or complex Hilbert spaces.

2 Definition of Iso-inner product and properties

Let H be a real or complex Hilbert space with an inner product $(\cdot,\cdot)$.

We suppose

(H1) $\hat{T} \in \mathcal{L}(H)$, $\hat{T}^{-1}$ exists and $\hat{T}^{-1} \in \mathcal{L}(H)$ and $\hat{T}^{-1}$ is positive, i.e. $\hat{T}^{-1} : H \longrightarrow H$ is self-adjoint operator and $(\hat{T}^{-1}x, x) > 0$ for every $x \in H$

We lift the Hilbert space H into the set

$$\hat{H} := \{\hat{T}^{-1}(x) := \hat{x} \quad \text{for} \quad x \in H\}$$

and we define so called iso-inner product as follows

$$\widehat{(\hat{x}, \hat{y})} := (\hat{T}^{-1}x, \hat{T}^{-1}y)\frac{1}{\hat{T}_1} \quad \text{for} \quad \hat{x}, \hat{y} \in \hat{H}.$$

Since $\hat{T}^{-1} \in \mathcal{L}(H)$ then $\hat{H}$ is a linear space.

Remark 2.1. *We note that the initial space H should be Hilbert space because the term positive definitely operator we can define on Hilbert space not on arbitrary linear space with inner product. The positive definiteness of $\hat{T}$ is one of the main assumption for it.*

Proposition 2.2. *We suppose $(H1)$. $\widehat{(\hat{\cdot}, \hat{\cdot})}$ satisfies all axioms for inner product.*

Proof. 1. Let $\hat{x} \in \hat{H}$ be arbitrary chosen.

$$\widehat{(\hat{x}, \hat{x})} = (\hat{T}^{-1}x, \hat{T}^{-1}x)\frac{1}{\hat{T}_1} \geq 0$$

because $\hat{T}_1 > 0$, $\hat{T}^{-1} : H \longrightarrow H$, $(\cdot, \cdot)$ is an inner product in the Hilbert space H.

Also,

$$\widehat{(\hat{x}, \hat{x})} = 0 \iff (\hat{T}^{-1}x, \hat{T}^{-1}x)\tfrac{1}{\hat{T}_1} \iff$$

$$(\hat{T}^{-1}x, \hat{T}^{-1}x) = 0 \iff \hat{T}^{-1}x = 0 \iff x = 0$$

because $\hat{T}^{-1} \in \mathcal{L}(H)$.

2. Let $\hat{\lambda} \in \hat{F}_{\mathbb{C}}$. Then for $\hat{x}, \hat{y} \in \hat{H}$ we have

$$\widehat{(\hat{\lambda}\hat{\times}\hat{x}, \hat{y})} = \left(\lambda\tfrac{1}{\hat{T}_1}T_1\hat{T}^{-1}x, \hat{T}^{-1}y\right)\tfrac{1}{\hat{T}_1}$$

$$= \lambda\tfrac{1}{\hat{T}_1}\hat{T}_1(\hat{T}^{-1}x, \hat{T}^{1}y)\tfrac{1}{\hat{T}_1}$$

$$= \hat{\lambda}\hat{\times}\widehat{(\hat{x}, \hat{y})}$$

and

$$\widehat{(\hat{x}, \hat{\lambda}\hat{\times}\hat{y})} = \left(\hat{T}^{-1}x, \lambda\tfrac{1}{\hat{T}_1}\hat{T}_1\hat{T}^{-1}y\right)\tfrac{1}{\hat{T}_1}$$

$$= \overline{\lambda\tfrac{1}{\hat{T}_1}\hat{T}_1}(\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1}$$

$$= \overline{\lambda}(\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{T_1}$$

$$= \overline{\lambda}\tfrac{1}{\hat{T}_1}T_1(\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1}$$

$$= \hat{\overline{\lambda}}\hat{\times}\widehat{(\hat{x}, \hat{y})}.$$

3. For $\hat{x}, \hat{y}, \hat{z} \in \hat{H}$ we have

$$\widehat{(\hat{x} + \hat{y}, \hat{z})} = (\hat{T}^{-1}x + \hat{T}^{-1}y, \hat{T}^{-1}z)\tfrac{1}{\hat{T}_1}$$

$$= [(\hat{T}^{-1}x, \hat{T}^{-1}z) + (\hat{T}^{-1}y, \hat{T}^{-1}z)]\tfrac{1}{\hat{T}_1}$$

$$= (\hat{T}^{-1}x, \hat{T}^{-1}z)\tfrac{1}{\hat{T}_1} + (\hat{T}^{-1}y, \hat{T}^{-1}z)\tfrac{1}{\hat{T}_1}$$

$$= \widehat{(\hat{x}, \hat{z})} + \widehat{(\hat{y}, \hat{z})}.$$

□

Remark 2.3. *We will note that not for any $\hat{T}$ we can make lift of a space with inner product in an iso-space with iso - inner product.*

Really, let us consider $\mathcal{C}([1,2])$ in which is defined inner product as follows

$$(f, g) = \int_1^2 f(x)g(x)dx \quad \text{for} \quad f, g \in \mathcal{C}([1,2]).$$

If we make the lift, then the corresponding iso-inner product is

$$\widehat{(\hat{f}, \hat{g})} = \hat{\int}_1^2 \hat{f}(\hat{x})\hat{\times}\hat{g}(\hat{x})\hat{\times}\hat{d}\hat{x}$$

$$= \int_1^2 f(x)g(x)\left(1 - x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\right)dx$$

and if $f = g$ *we have*

$$\widehat{(\hat{f},\hat{f})} = \int_1^2 f^2(x)\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

If $\hat{T}(x) = x + 1$ *then* $\hat{T}(x) > 0$ *for every* $x \in [1,2]$ *and we always have*

$$\widehat{(\hat{f},\hat{f})} = \int_1^2 f^2(x)\Big(1 - x\frac{1}{x+1}\Big)dx = \int_1^2 f^2(x)\frac{1}{x+1}dx \geq 0.$$

But if $\hat{T}(x) = e^x$, $x \in [1,2]$, *then* $\hat{T} > 0$ *for every* $x \in [1,2]$,**but we always have**

$$\widehat{(\hat{f},\hat{f})} = \int_1^2 f^2(x)(1-x)dx \leq 0$$

because $1 - x \leq 0$ *for every* $x \in [1,2]$, **i.e. not for every positive** $\hat{T}$ **we can define iso-inner iso - product.**

Consequently, we have to be careful when we choose the isotopic element $\hat{T}$, *in other words when we choose the isotopic element* $\hat{T}$ *we have to follow the assumptions in the beginning of this section.*

Examples 2.4. 1. *Let us consider* $\mathbb{R}^n$ *and let* $\hat{T} = (\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_n)$, *where* $\hat{T}_l$, $l = 1,2,\ldots,n$ *are positive reals. Then we lift* $\mathbb{R}^n$ *into the space* $\hat{R}^n$ *in the following manner: for given* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *the corresponding iso-element* $\hat{x}$ *will be*

$$\hat{x} = \Big(\frac{x_1}{\hat{T}_1}, \frac{x_2}{\hat{T}_2}, \ldots, \frac{x_n}{\hat{T}_n}\Big)$$

and for $\hat{x}$, $\hat{y} \in \hat{\mathbb{R}}^n$ *we define iso-inner iso product as follows*

$$\widehat{(\hat{x},\hat{y})} = \sum_{l=1}^n \hat{x}_l \hat{\times} \hat{y}_l = \sum_{l=1}^n x_l \tfrac{1}{\hat{T}_l}\hat{T}_l y_l \tfrac{1}{\hat{T}_l} = \sum_{l=1}^n x_l y_l \tfrac{1}{\hat{T}_l}.$$

2. *The space* l_2 *which is consisted of all real sequences* $\xi = \{\xi_l\}_{l=1}^\infty$ *so that* $\sum_{l=1}^\infty \xi_l^2 < \infty$. *Let also* $\hat{T} = \{\hat{T}_l\}_{l=1}^\infty$ *to be a sequence of positive reals. To be made suitable lift of* l_2 *into* $\hat{l}_2$ *as follows*

$$\xi \longrightarrow \hat{\xi} = \{\hat{\xi}_l\}_{l=1}^\infty = \Big\{\frac{\xi_l}{\hat{T}_l}\Big\}_{l=1}^\infty$$

we have a need of an additional condition for $\hat{T}$, *namely the sequence* $\{\hat{T}_l\}_{l=1}^{\infty}$ *to be bounded below of a positive real* a. *In this way we have*

$$\frac{1}{\hat{T}_l} \leq \frac{1}{a} \quad \text{for} \quad \forall l = 1, 2, \ldots$$

and from $\xi \in l_2$ *it follows*

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} \leq \frac{1}{a^2} \sum_{l=1}^{\infty} \xi_l^2 < \infty.$$

In this case we define iso-inner iso-product in $\hat{l}_2$ *in the following manner: for* $\hat{\xi}$, $\hat{\eta} \in \hat{l}_2$

$$\widehat{(\hat{\xi}, \hat{\eta})} = \sum_{l=1}^{\infty} \hat{\xi}_l \hat{\times} \hat{\eta}_l = \sum_{l=1}^{\infty} \xi_l \frac{1}{\hat{T}_l} \hat{T}_l \eta_l \frac{1}{\hat{T}_l} = \sum_{l=1}^{\infty} \xi_l \eta_l \frac{1}{\hat{T}_l}.$$

The condition the sequence $\{\hat{T}_l\}_{l=1}^{\infty}$ *to be bounded below is very important. Really, let* $\xi = \{\frac{1}{l}\}_{l=1}^{\infty}$ *and* $\hat{T} = \{\frac{1}{\sqrt{l}}\}_{l=1}^{\infty}$. *Then*

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty,$$

the sequence $\hat{T}$ *is not bounded below of positive real and*

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{1}{l^2} l = \sum_{l=1}^{\infty} \frac{1}{l} = \infty.$$

So far we obtain if the positive sequence $\hat{T}$ *is bounded below of positive real and* $\xi \in l^2$ *then* $\hat{\xi} \in \hat{l}_2$. *The inverse is not valid in every case. Really, let* $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty} = \{l\}_{l=1}^{\infty}$. *Then* $\hat{T}$ *is a bounded below sequence from* 1. *Let also,* $\xi = \{\xi_l\}_{l=1}^{\infty} = \left\{\frac{1}{\sqrt{l}}\right\}_{l=1}^{\infty}$. *Then*

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{1}{l} = \infty,$$

i.e. $\xi \notin l_2$. *But for*

$$\hat{\xi} = \left\{\frac{\xi_l}{\hat{T}_l}\right\}_{l=1}^{\infty} = \left\{\frac{1}{l^{\frac{3}{2}}}\right\}_{l=1}^{\infty}$$

we have

$$\sum_{l=1}^{\infty} \hat{\xi}_l^2 = \sum_{l=1}^{\infty} \frac{1}{l^3} < \infty.$$

This example shows that we have $\hat{\xi} \in \hat{l}_2$ *and* $\xi \notin l_2$.

Now we will give a condition for the positive sequence $\hat{T}$ *such that from* $\hat{\xi} \in \hat{l}_2$ *it follows* $\xi \in l^2$. *We suppose that the positive sequence* $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty}$ *is a bounded above of the positive real* b. *For* $\hat{\xi} = \left\{\frac{\xi_l}{\hat{T}_l}\right\}_{l=1}^{\infty} \in \hat{l}_2$ *we have*

$$\frac{1}{b^2}\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{\xi_l^2}{b^2} \leq \sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} < \infty$$

and since $b > 0$ *we conclude that*

$$\sum_{l=1}^{\infty} \xi_l^2 < \infty,$$

i.e. $\xi = \{\xi_l\}_{l=1}^{\infty} \in l_2$.

If the positive sequence $\hat{T}$ *is bounded above from the positive real* b *from* $\xi \in l^2$ *does not follow that* $\hat{\xi} \in \hat{l}_2$. *Really, let* $\xi = \left\{\frac{1}{l}\right\}_{l=1}^{\infty}$. *Then*

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty,$$

i.e. $\xi \in l_2$. *Let now* $\hat{T} = \left\{\frac{1}{\sqrt{l}}\right\}_{l=1}^{\infty}$. *Then* $\hat{T}$ *is a bounded above sequence from* 1 *and*

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{\frac{1}{l^2}}{\frac{1}{l}} = \sum_{l=1}^{\infty} \frac{1}{l} = \infty,$$

consequently $\hat{\xi} = \left\{\frac{\xi_l}{\hat{T}_l}\right\}_{l=1}^{\infty} \notin \hat{l}_2$.

The condition the sequence $\hat{T}$ *to be bounded above is too important. Indeed, let* $\hat{T} = \{l^2\}_{l=1}^{\infty}$ *and* $\xi = \{\sqrt{l}\}_{l=1}^{\infty}$. *Then* $\hat{T}$ *is unbounded above and*

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{l}{l^4} = \sum_{l=1}^{\infty} \frac{1}{l^3} < \infty,$$

in other words $\hat{\xi} \in \hat{l}_2$, *and*

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} l = \infty,$$

consequently $\xi \notin l_2$.

If we suppose that the positive sequence $\hat{T}$ *is bounded below and above from the positive reals* a *and* b, *respectively, then from* $\xi \in l^2$ *it follows that* $\hat{\xi} \in \hat{l}_2$ *and from* $\hat{\xi} \in \hat{l}_2$ *it follows that* $\xi \in l_2$, *because*

$$\frac{1}{b^2} \sum_{l=1}^{\infty} \xi_l^2 \leq \sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} \leq \frac{1}{a^2} \sum_{l=1}^{\infty} \xi_l^2.$$

Remark 2.5. *In the above example we saw that if* $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty}$ *is a bounded below positive sequence from positive real then there exists* $\hat{\xi} \in \hat{l}_2$ *so that* $\xi \notin l_2$. *Now we will see that if* $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty}$ *is a bounded below positive sequence from positive real then there exists* $\hat{\xi} \in \hat{l}_2$ *such that* $\xi \in l_2$. *Let*

$$\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty} = \{l\}_{l=1}^{\infty}$$

which is bounded below from 1, *and let*

$$\xi = \{\xi_l\}_{l=1}^{\infty} = \left\{\frac{1}{l^2}\right\}_{l=1}^{\infty}.$$

Then

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{\frac{1}{l^4}}{l^2} = \sum_{l=1}^{\infty} \frac{1}{l^6} < \infty, \quad \text{i.e.} \quad \hat{\xi} \in \hat{l}_2,$$

and

$$\sum_{l=1}^{\infty}\xi_l^2=\sum_{l=1}^{\infty}\frac{1}{l^4}<\infty,\quad \text{i.e.}\quad \xi\in l_2.$$

Remark 2.6. *In the above example we saw that if* $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ *is a bounded above positive sequence then there exists* $\xi\in l_2$ *so that* $\hat{\xi}\notin\hat{l}_2$. *Now we will see that if* $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ *is a bounded above positive sequence then there exists* $\xi\in l_2$ *so that* $\hat{\xi}\in\hat{l}_2$. *Let*

$$\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}=\left\{\frac{1}{\sqrt{l}}\right\}_{l=1}^{\infty}$$

which is bounded above from 1, *and let*

$$\xi=\{\xi_l\}_{l=1}^{\infty}=\left\{\frac{1}{l^4}\right\}_{l=1}^{\infty}.$$

Then

$$\sum_{lk=1}^{\infty}\xi_l^2=\sum_{l=1}^{\infty}\frac{1}{l^8}<\infty,\quad \text{i.e.}\quad \xi\in l_2,$$

$$\sum_{l=1}^{\infty}\frac{\xi_l^2}{\hat{T}_l^2}=\sum_{l=1}^{\infty}\frac{\frac{1}{l^8}}{\frac{1}{l}}=\sum_{l=1}^{\infty}\frac{1}{l^7}<\infty,\quad \text{i.e.}\quad \hat{\xi}\in\hat{l}_2.$$

From the above examples and remarks follow that there exists an iso-space which is generalization of the iso-space $\hat{l}_2$. Now we will construct it.

Definition 2.7. *For given sequence* $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ *of positive reals we define the iso-space*

$$\hat{ll}_{\hat{T}}=\left\{\frac{\xi_l}{\hat{T}_l}:\xi_l\in\mathbb{R}^+,\sum_{l=1}^{\infty}\frac{\xi_l^2}{\hat{T}_l^2}<\infty\right\}.$$

With $\hat{\mathcal{T}}$ we will denote the set of all sequences of positive reals.

Definition 2.8. *The set*

$$\hat{ll}_2=\cup_{\hat{T}\in\hat{\mathcal{T}}}\hat{ll}_{\hat{T}}$$

will be called iso-generalized iso - space of $\hat{l}_2$.

From above investigations we have the following inclusion

Proposition 2.9. *$\hat{l}_2 \subset \hat{ll}_2$ and $\hat{l}_2 \neq \hat{ll}_2$.*

Proposition 2.10. *(iso-Cauchy inequality) We suppose $(H1)$. For $\hat{x}$, $\hat{y} \in \hat{H}$ we have*

$$\widehat{(\hat{x},\hat{y})}\hat{\times}\widehat{(\hat{x},\hat{y})} \leq \widehat{(\hat{x},\hat{x})}\hat{\times}\widehat{(\hat{y},\hat{y})}.$$

Proof. Let $\hat{\lambda} \in \hat{F}_{\mathbb{C}}$. Then

$$A := (\hat{x} + \hat{\lambda}\hat{\times}\hat{y}, \hat{x} + \hat{\lambda}\hat{\times}\hat{y})$$

$$= (\hat{x}, \widehat{\hat{x} + \hat{\lambda}\hat{\times}\hat{y}}) + (\hat{\lambda}\hat{\times}\hat{y}, \widehat{\hat{x} + \hat{\lambda}\hat{\times}\hat{y}})$$

$$= \widehat{(\hat{x},\hat{x})} + \hat{\bar{\lambda}}\hat{\times}\widehat{(\hat{x},\hat{y})} + \hat{\lambda}\hat{\times}\widehat{(\hat{y},\hat{x})} + \hat{\lambda}\hat{\times}\hat{\bar{\lambda}}\hat{\times}\widehat{(\hat{y},\hat{y})} \geq 0.$$

Let

$$\hat{\lambda} = -\widehat{(\hat{x},\hat{y})} \prec \widehat{(\hat{y},\hat{y})}.$$

Then

$$A = \widehat{(\hat{x},\hat{x})} - \widehat{(\hat{x},\hat{y})}\hat{\times}\widehat{(\hat{x},\hat{y})} \prec \widehat{(\hat{y},\hat{y})} - \widehat{(\hat{x},\hat{y})}\hat{\times}\widehat{\widehat{(\hat{x},\hat{y})}} \prec \widehat{(\hat{y},\hat{y})}$$

$$+\widehat{(\hat{x},\hat{y})}\hat{\times}\widehat{\widehat{(\hat{x},\hat{y})}}\hat{\times}\widehat{(\hat{y},\hat{y})} \prec (\widehat{(\hat{y},\hat{y})}\hat{\times}\widehat{(\hat{y},\hat{y})})$$

and since $A \geq 0$ we get

$$\widehat{(\hat{x},\hat{y})} - \widehat{\widehat{(\hat{x},\hat{y})}}\hat{\times}\widehat{(\hat{x},\hat{y})} \prec \widehat{(\hat{y},\hat{y})} \geq 0,$$

from where we obtain the assertion of the statement. □

Definition 2.11. *Two elements $\hat{x}$, $\hat{y} \in \hat{H}$ will be called iso-orthogonal if*

$$\widehat{(\hat{x},\hat{y})} = 0.$$

Remark 2.12. *We will note that if two elements of the Hilbert space H are orthogonal with respect the inner product $(\cdot,\cdot)$ they can not be iso-orthogonal iso-elements of the iso-space $\hat{H}$ with respect the iso-inner iso-product $\widehat{(\hat{\cdot},\hat{\cdot})}$ and the inverse. We will consider an example for this.*

Let $H = \mathcal{C}([-1,1])$ with inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx.$$

Then x, x^2, $x^3 \in H$ and

$$(x,x^2) = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{x=-1}^{x=1} = \frac{1}{4} - \frac{1}{4} = 0,$$

$$(x,x^3) = \int_{-1}^{1} x^4 dx = \frac{x^5}{5}\Big|_{x=-1}^{x=1} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$$

Consequently x and x^2 are orthogonal elements of $\mathcal{C}([-1,1])$ and x and x^3 are not orthogonal elements of $\mathcal{C}([-1,1])$.

Let now

$$\hat{T}(x) = e^{x+\frac{7}{10}x^2}, \quad x \in [-1,1].$$

Then

$$\hat{T}(x) \geq 0 \quad \text{for} \quad \forall x \in [-1,1]$$

and

$$\hat{T}'(x) = \Big(1 + \tfrac{7}{5}x\Big)e^{x+\frac{7}{10}x^2},$$

$$\frac{\hat{T}'(x)}{\hat{T}(x)} = \frac{\Big(1+\frac{7}{5}x\Big)e^{x+\frac{7}{5}x^2}}{e^{x+\frac{7}{5}x^2}} = 1 + \tfrac{7}{5}x,$$

$$1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} = 1 - x\Big(1 + \tfrac{7}{5}x\Big) = 1 - x - \tfrac{7}{5}x^2.$$

Now we consider $\hat{\mathcal{C}}([-1,1])$ with the isotopic element $\hat{T}(x)$, then the corresponding iso-inner iso- product is

$$\widehat{(\hat{f},\hat{g})} = \int_{-1}^{1} f(x)g(x)\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Then

$$\widehat{(\hat{x},\hat{x}^2)} = \int_{-1}^{1} x^3\Big(1 - x - \tfrac{7}{5}x^2\Big)dx$$

$$= \int_{-1}^{1} x^3 dx - \int_{-1}^{1} x^4 dx - \tfrac{7}{5}\int_{-1}^{1} x^5 dx$$

$$= \tfrac{x^4}{4}\Big|_{x=-1}^{x=1} - \tfrac{x^5}{5}\Big|_{x=-1}^{x=1} - \tfrac{7}{5}\tfrac{x^6}{6}\Big|_{x=-1}^{x=1}$$

$$= -\tfrac{2}{5} \neq 0,$$

and

$$\widehat{(\hat{x},\hat{x}^3)} = \int_{-1}^{1} x^4\Big(1 - x - \tfrac{7}{5}x^2\Big)dx$$

$$= \int_{-1}^{1} x^4 dx - \int_{-1}^{1} x^5 dx - \tfrac{7}{5}\int_{-1}^{1} x^6 dx$$

$$= \tfrac{x^5}{5}\Big|_{x=-1}^{x=1} - \tfrac{x^6}{6}\Big|_{x=-1}^{x=1} - \tfrac{7}{5}\tfrac{x^7}{7}\Big|_{x=-1}^{x=1}$$

$$= \tfrac{2}{5} - \tfrac{7}{5}\tfrac{2}{7} = 0.$$

Consequently $\hat{x}$ and $\hat{x}^2$ are not iso-orthogonal in $\hat{\mathcal{C}}([-1,1])$ and $\hat{x}$ and $\hat{x}^3$ are iso-orthogonal iso-element in $\hat{\mathcal{C}}([-1,1])$.

Below we will give a criteria from which if $x, y \in H$ are orthogonal then $\hat{x}$, $\hat{y} \in \hat{H}$ are iso-orthogonal and the inverse.

Proposition 2.13. *We suppose (H1) and $\hat{T} = \hat{T}^{-1*}$. If $x, y \in H$ are orthogonal then $\hat{x}$, $\hat{y} \in \hat{H}$ are iso-orthogonal and the inverse.*

Here with $\hat{T}^{-1*}$ we denote the adjoint operator of the operator $\hat{T}^{-1}$. Since $\hat{T}^{-1}$ is positive definite then $\hat{T}^{-1} = \hat{T}^{-1*}$ and from here $\hat{T} = \hat{T}^{-1*}$ is equivalent of $\hat{T}^2 = I$.

Proof. Below with I we will denote the identity operator in $\mathcal{L}(H)$.

1. Let $x, y \in H$ are orthogonal. Then

$$(x, y) = 0. \tag{2.1}$$

On the other hand we have

$$= \widehat{(\hat{x}, \hat{y})} = (\hat{T}^{-1}x, \hat{T}^{-1}y)\frac{1}{\hat{T}_1}$$

now we use that $\hat{T}^{-1*} = \hat{T}$

$$(\hat{T}^{-1*}\hat{T}^{-1}x, y)\frac{1}{\hat{T}_1} = (\hat{T}\hat{T}^{-1}x, y)\frac{1}{\hat{T}_1} = (x, y)\frac{1}{\hat{T}_1},$$

and using (2.1) we conclude that

$$\widehat{(\hat{x}, \hat{y})} = 0.$$

Consequently $\hat{x}$ and $\hat{y}$ are iso-orthogonal.

2. Let now $\hat{x}$, $\hat{y} \in \hat{H}$ are iso-orthogonal. Then

$$\widehat{(\hat{x}, \hat{y})} = 0. \tag{2.2}$$

On the other hand, using (2.2), we have

$$(x, y)\tfrac{1}{\hat{T}_1} = (Ix, y)\tfrac{1}{\hat{T}_1} = (\hat{T}\hat{T}^{-1}x, y)\tfrac{1}{\hat{T}_1}$$

$$= (\hat{T}^{-1*}\hat{T}^{-1}x, y)\tfrac{1}{\hat{T}_1} = (\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1} = 0.$$

Therefore x and y are orthogonal elements in H.

□

Proposition 2.14. *We suppose* $(H1)$ *and* $\hat{T}^{-1*} = \hat{T}$. *Then in* $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \in \hat{H}$ *is an iso-orthogonal system then it is iso-linear independent system.*

Proof. Since $\hat{T}^{-1*} = \hat{T}$ from the previous Proposition follows that the system $x_1, x_2, \ldots, x_n \in H$ is an orthogonal system. From the properties of the linear spaces with an inner product we conclude that $x_1, x_2, \ldots, x_n$ is

linear independent system in H. From here and from, since $\hat{T}$ is a linear operator,

$$\hat{\lambda}_1 \hat{\times} \hat{x_1} + \hat{\lambda}_2 \hat{\times} \hat{x_2} + \cdots + \hat{\lambda}_n \hat{\times} \hat{x}_n = 0 \quad \Longleftrightarrow$$

$$\lambda_1 \hat{T}^{-1} x_1 + \lambda_2 \hat{T}^{-1} x_2 + \cdots + \lambda_n \hat{T}^{-1} x_n = 0 \quad \Longleftrightarrow$$

$$\hat{T}(\lambda_1 \hat{T}^{-1} x_1 + \lambda_2 \hat{T}^{-1} x_2 + \cdots + \lambda_n \hat{T}^{-1} x_n) = \hat{T}0 \quad \Longleftrightarrow$$

$$\hat{T}(\lambda_1 \hat{T}^{-1} x_1) + \hat{T}(\lambda_2 \hat{T}^{-1} x_2) + \cdots + \hat{T}(\lambda_n \hat{T}^{-1} x_n) = 0 \quad \Longleftrightarrow$$

$$\lambda_1 \hat{T}\hat{T}^{-1} x_1 + \lambda_2 \hat{T}\hat{T}^{-1} x_2 + \cdots + \lambda_n \hat{T}\hat{T}^{-1} x_n = 0 \quad \Longleftrightarrow$$

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0.$$

we conclude that the system $\hat{x}_1$, $\hat{x}_2$, ..., $\hat{x}_n$ are iso-linear independent. □

Proposition 2.15. *We suppose* $(H1)$. *Let* $\hat{x}$, $\hat{y} \in \hat{H}$ *are iso-orthogonal. Then*

$$\widehat{(\hat{x} + \hat{y}, \hat{x} + \hat{y})} = \widehat{(\hat{x}, \hat{x})} + \widehat{(\hat{y}, \hat{y})}.$$

Proof. Since $\hat{x}$ and $\hat{y}$ are iso-orthogonal then

$$\widehat{(\hat{x}, \hat{y})} = \widehat{(\hat{y}, \hat{x})} = 0.$$

From here

$$\widehat{(\hat{x} + \hat{y}, \hat{x} + \hat{y})} = \widehat{(\hat{x}, \hat{x} + \hat{y})} + \widehat{(\hat{y}, \hat{x} + \hat{y})}$$

$$\widehat{(\hat{x}, \hat{x})} + \widehat{(\hat{x}, \hat{y})} + \widehat{(\hat{y}, \hat{x})} + \widehat{(\hat{y}, \hat{y})}$$

$$= \widehat{(\hat{x}, \hat{x})} + \widehat{(\hat{y}, \hat{y})}.$$

□

Proposition 2.16. *We suppose* $(H1)$. *If* $\hat{x}$, $\hat{y} \in \hat{H}$ *then*

$$\widehat{(\hat{x} + \hat{y}, \hat{x} + \hat{y})} + \widehat{(\hat{x} - \hat{y}, \hat{x} - \hat{y})} = \hat{2} \hat{\times} \widehat{(\hat{x}, \hat{x})} + \hat{2} \hat{\times} \widehat{(\hat{y}, \hat{y})}.$$

Proof.

$$(\hat{x}+\widehat{\hat{y},\hat{x}}+\hat{y})+(\hat{x}-\widehat{\hat{y},\hat{x}}-\hat{y})$$

$$=(\widehat{\hat{x},\hat{x}+\hat{y}})+(\widehat{\hat{y},\hat{x}+\hat{y}})+(\widehat{\hat{x},\hat{x}-\hat{y}})-(\widehat{\hat{y},\hat{x}-\hat{y}})$$

$$=\widehat{(\hat{x},\hat{x})}+\widehat{(\hat{x},\hat{y})}+\widehat{(\hat{y},\hat{x})}+\widehat{(\hat{y},\hat{y})}+\widehat{(\hat{x},\hat{x})}-\widehat{(\hat{x},\hat{y})}-\widehat{(\hat{y},\hat{x})}+\widehat{(\hat{y},\hat{y})}$$

$$=2\widehat{(\hat{x},\hat{x})}+2\widehat{(\hat{y},\hat{y})}$$

$$=\hat{2}\hat{\times}\widehat{(\hat{x},\hat{x})}+\hat{2}\hat{\times}\widehat{(\hat{y},\hat{y})}.$$

□

Definition 2.17. *We suppose* $(H1)$. *We will say that the iso-sequence* $\{\hat{x}_n\}_{n=1}^{\infty}$ *of iso-elements of* $\hat{H}$ *is convergent to* $\hat{x}\in\hat{H}$ *if*

$$\lim_{n\longrightarrow\infty}(\widehat{\hat{x}_n,\hat{x}_n})=\lim_{n\longrightarrow\infty}(\hat{T}^{-1}x_n,\hat{T}^{-1}x_n)\frac{1}{\hat{T}_1}$$

$$=(\hat{T}^{-1}x,\hat{T}^{-1}x)\frac{1}{\hat{T}_1}$$

$$=\widehat{(\hat{x},\hat{x})}.$$

Remark 2.18. *We suppose* $(H1)$. *If* $x_n\in H$ *and* $\lim_{n\longrightarrow\infty}x_n=x$ *in* H, *since* $\hat{T}^{-1}\in\mathcal{L}(H)$ *we have that*

$$\lim_{n\longrightarrow\infty}\hat{T}^{-1}x_n=\hat{T}^{-1}x\quad\text{in}\quad H$$

or

$$\lim_{n\longrightarrow\infty}(\hat{T}^{-1}x_n,\hat{T}^{-1}x_n)=(\hat{T}^{-1}x,\hat{T}^{-1}x)$$

and from the definition above we obtain that

$$\lim_{n\longrightarrow\infty}(\widehat{\hat{x}_n,\hat{x}_n})=\widehat{(\hat{x},\hat{x})}.$$

If $\hat{x}_n\longrightarrow_{n\longrightarrow\infty}\hat{x}$ *then from the definition above*

$$\hat{T}^{-1}x_n\longrightarrow_{n\longrightarrow\infty}\hat{T}^{-1}x\quad\text{in}\quad H$$

and since $\hat{T}^{-1} \in \mathcal{L}(H)$ *then* $x_n \longrightarrow_{n\longrightarrow\infty} x$ *in* H.

Consequently, under the assumption $(H1)$ *the convergence in* H *and* $\hat{H}$ *are equivalent.*

Proposition 2.19. *We suppose* $(H1)$. *If* $\hat{x}_n, y \in \hat{H}$, $\lim_{n\longrightarrow\infty} \hat{x}_n = \hat{x} \in \hat{H}$, *then*

$$\lim_{n\longrightarrow\infty} \widehat{(\hat{x}_n, \hat{y})} = \widehat{(\hat{x}, \hat{y})}.$$

Proof. Since the convergence in H and $\hat{H}$ are equivalent, then $\lim_{n\longrightarrow\infty} x_n = x$. From $\hat{T}^{-1} \in \mathcal{L}(H)$ we conclude that

$$\lim_{n\longrightarrow\infty} \hat{T}^{-1}x_n = \hat{T}^{-1}x.$$

From here and from the continuous property of the inner product in H we get

$$\lim_{n\longrightarrow\infty} (\hat{T}^{-1}x_n, \hat{T}^{-1}y) = (\hat{T}^{-1}x, \hat{T}^{-1}y)$$

from where we conclude

$$\lim_{n\longrightarrow\infty} \widehat{(\hat{x}_n, y)} = \widehat{(\hat{x}, \hat{y})}.$$

□

Definition 2.20. *We suppose* $(H1)$. *Let* $\hat{x}_n \in \hat{H}$ *is convergent sequence in* $\hat{H}$ *to the element* $\hat{x} \in \hat{H}$. *Then since the convergence in* H *and in* $\hat{H}$ *are equivalent we have that* x_n *is convergent sequence in* H *and* $x \in H$, *because* H *is Hilbert space. Because* $\hat{T}^{-1} \in \mathcal{L}(H)$ *we have that*

$$\hat{T}^{-1}x_n \longrightarrow_{n\longrightarrow\infty} \hat{T}^{-1}x$$

and $\hat{T}^{-1}x \in \hat{H}$. *Therefore* $\hat{H}$ *is a completely space which will be called iso-Hilbert iso-space.*

Definition 2.21. *We suppose* $(H1)$. *Then if* $M \subset H$ *we will write* $\hat{M} \subset \hat{H}$.

Since $\hat{T}^{-1} \in \mathcal{L}(H)$ *then if* M *is a closed subset of* H *we have that* $\hat{M}$ *is a closed subset of* $\hat{H}$ *and if* M *is a convex subset of* H *then* $\hat{M}$ *is a convex subset of* $\hat{H}$.

If $\hat{M}$ is a subspace of the iso-Hilbert iso-space then every $\hat{x} \in \hat{H}$ can be represented in the form

$$\hat{x} = \hat{y} + \hat{z}$$

where $\hat{y} \in \hat{M}$ and $\hat{z}$ is an iso-orthogonal of every iso-element of $\hat{M}$.

If $\hat{M} \subset \hat{H}$ is a linear manifold then the iso-set of all iso-elements of $\hat{H}$ which are iso-orthogonal of every iso-element of $\hat{M}$ will be called iso-orthogonal suplement of $\hat{M}$ and will be denoted by $\hat{M}^{\perp}$.

3 Iso-operators in iso-Hilbert spaces

Definition 3.1. *We suppose $(H1)$.*

Let $A : H \longrightarrow H$ is a linear operator. The corresponding lift of A will be defined as follows

$$\hat{A} = A\hat{T}^{-1}$$

and will be called iso-operator.

The act of the iso-operator $\hat{A}$ on the iso-element $\hat{x} \in \hat{H}$ will be defined as follows

$$\hat{A}\hat{x} := \hat{A}\hat{T}^{-1}\hat{T}\hat{T}^{-1}x = \hat{A}\hat{T}^{-1}x.$$

Because A and $\hat{T}^{-1}$ are linear operators in $\mathcal{L}(H)$ and since the composition of two linear operators is a linear operator we have that $\hat{A}$ is a linear operator.

Iso-norm of the iso-operator $\hat{A}$ will be defined as follows

$$\widehat{||\hat{A}||} := ||A\hat{T}^{-1}||\frac{1}{\hat{T}_1} = \frac{1}{\hat{T}_1}\sup_{x\in H:||x||\leq 1}||A\hat{T}^{-1}x||.$$

The iso-operator $\hat{A}$ will be called iso-bounded if there exists $\hat{c} \in \hat{F}_{\mathbb{R}}$, $\hat{c} \geq 0$, such that

$$\widehat{||\hat{A}||} \leq \hat{c}.$$

Remark 3.2. *If $A : H \longrightarrow H$ is a linear bounded operator then $\hat{A}$ is an iso-bounded linear iso-operator. Really, since $A, \hat{T}^{-1} \in \mathcal{L}(\mathcal{H})$ then there exists a constant $c > 0$ such that*

$$||A|| \leq c, \quad ||T^{-1}|| \leq c.$$

From here follows

$$\widehat{||\hat{A}||} = ||A\hat{T}^{-1}||\frac{1}{\hat{T}_1} \leq ||A||||\hat{T}^{-1}||\frac{1}{\hat{T}_1} \leq \frac{c^2}{\hat{T}_1} < \infty.$$

Remark 3.3. *If $\hat{A}$ is iso-bounded linear iso-operator then there is a possibility A to be unbounded operator. To see this we will consider the following example.*

Let $\mathcal{C}^+([0,1])$ to be the space of all nonnegative continuous functions on $[0,1]$, endowed with standard maximum norm, and let

$$Af(t) = \int_0^1 \tfrac{1}{s} f(s)ds,$$

$$\hat{T}^{-1}f(t) = \int_0^t sf(s)ds, \quad t \in [0,1], f \in \mathcal{C}([0,1]).$$

Then A is unbounded operator, because when $f \equiv 1$ we have

$$A1 = \int_0^1 \tfrac{1}{s} ds = -\ln 0,$$

$$|A1| = |\ln 0| = \infty.$$

from where, since $||1|| = 1$ and $||A|| = \sup_{x \in \mathcal{C}^+([0,1]), ||x|| \leq 1} ||Ax||$ we conclude that A is unbounded operator.

But, for arbitrary $f \in \mathcal{C}^+([0,1])$, we have

$$A\hat{T}^{-1}f(x) = \int_0^1 \tfrac{1}{s}\hat{T}^{-1}f(s)ds$$

$$= \int_0^1 \tfrac{1}{s}\int_0^s s_1 f(s_1)ds_1 ds.$$

From here

$$|A\hat{T}^{-1}f(x)| = \left|\int_0^1 \tfrac{1}{s}\int_0^s s_1 f(s_1)ds_1 ds\right|$$

$$= \int_0^1 \tfrac{1}{s}\int_0^s s_1 f(s_1)ds_1 ds$$

$$\leq ||f|| \int_0^1 \tfrac{1}{s}\int_0^s s_1 ds_1 ds$$

$$= ||f||\tfrac{1}{2}\int_0^1 \tfrac{1}{s}s^2 ds$$

$$= ||f||\tfrac{1}{2}\int_0^1 sds$$

$$= ||f||\tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}||f||,$$

from where

$$||A\hat{T}^{-1}f|| \leq \frac{1}{4}||f||$$

and therefore

$$||A\hat{T}^{-1}|| \leq \frac{1}{4}$$

and then

$$||A\hat{T}^{-1}||\frac{1}{\hat{T}_1} \leq \frac{1}{4}\frac{1}{\hat{T}_1} \quad \Longleftrightarrow \widehat{||\hat{A}||} \leq \frac{\hat{1}}{4},$$

i.e. $\hat{A}$ is iso-bounded iso-operator.

Definition 3.4. *The linear iso-operator $\hat{A} : \hat{H} \longrightarrow \hat{H}$ will be called iso-continuous iso-operator at $\hat{x_0} \in \hat{H}$ if from $\hat{x_n} \in \hat{H}$ and*

$$\widehat{||\hat{x_n} - \hat{x_0}||} \longrightarrow_{n\longrightarrow\infty} 0$$

it follows

$$\widehat{||\hat{A}\hat{x_n} - \hat{A}\hat{x_0}||} \longrightarrow_{n\longrightarrow\infty} 0.$$

The linear iso-operator $\hat{A} : \hat{H} \longrightarrow \hat{H}$ will be called iso-continuous in $\hat{H}$ if it is iso-continuous at every iso-point of $\hat{H}$.

Theorem 3.5. *The iso-bound and iso-continuity properties of the linear iso-operators are equivalent.*

Proof. Let $\hat{A} : \hat{H} \longrightarrow \hat{H}$ is linear iso-operator.

1. We suppose that $\hat{A}$ is an iso-bounded iso-operator. Then there exists $\hat{c} \in \hat{F}_R$ such that
$$\widehat{||\hat{A}\hat{x}||} \leq \hat{c}\hat{\times}\widehat{||\hat{x}||}.$$
Since $\hat{A}$ is a linear iso-operator we have
$$||\hat{A}\widehat{(\hat{x}_n - \hat{x})}|| = ||\hat{A}\widehat{\hat{x}_n - \hat{A}\hat{x}}|| \leq \hat{c}\hat{\times}||\widehat{\hat{x}_n - \hat{x}}||,$$
therefore from
$$||\widehat{\hat{x}_n - \hat{x}}|| \longrightarrow_{n\longrightarrow\infty} 0$$
it follows
$$||\hat{A}\hat{x_n} - \widehat{\hat{A}\hat{x}}|| \longrightarrow_{n\longrightarrow\infty} 0.$$
Consequently $\hat{A} : \hat{H} \longrightarrow \hat{H}$ is iso-continuous iso-operator.

2. We suppose that $\hat{A} : \hat{H} \longrightarrow \hat{H}$ is iso-continuous and iso-unbounded. Therefore from $\hat{x}_n \in \hat{H}$, $\widehat{||\hat{x}_n||} \leq \hat{I}$ we have that
$$\widehat{||\hat{A}\hat{x}_n||} \geq \hat{n}.$$
If we put $\hat{x}'_n = \hat{x}_n \curlywedge \hat{n}$ then
$$\widehat{||\hat{A}\hat{x}'_n||} = \hat{I} \curlywedge \hat{n}\hat{\times}\widehat{||\hat{A}\hat{x}_n||} \geq \hat{I} \tag{3.1}$$
and
$$\widehat{||\hat{x}'_n||} = \hat{I} \curlywedge \hat{n}\hat{\times}\widehat{||\hat{x}_n||} \leq \hat{I} \curlywedge \hat{n} \longrightarrow_{n\longrightarrow 0} 0$$
and since $\hat{A}$ is iso-continuous then
$$\widehat{||\hat{A}\hat{x}'_n||} \longrightarrow_{n\longrightarrow\infty} 0,$$
which contradicts with (3.1).

□

Remark 3.6. *Since, as we saw that if $\hat{A}$ is an iso-bounded iso-operator then in the general case we have not that A is a bounded operator and because iso-bounded and iso-continuity properties are equivalent then from $\hat{A}$ iso-continuous iso-operator in the general case does not follow that A is continuous operator.*

Definition 3.7. *The space of all linear iso-bounded iso-operators acting from $\hat{H}$ in $\hat{H}$ will be denoted with $\hat{\mathcal{L}}(\hat{H})$.*

Definition 3.8. *If $\hat{A}$, $\hat{B} \in \hat{\mathcal{L}}(\hat{H})$ then their composition is defined as follows*

$$\hat{A}\hat{B} = A\hat{T}^{-1}\hat{T}\hat{B}\hat{T}^{-1} = AB\hat{T}^{-1},$$

and

$$\hat{A}^2 = \hat{A}\hat{A} = A\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^2\hat{T}^{-1},$$

$$\hat{A}^3 = \hat{A}\hat{A}^2 = A\hat{T}^{-1}\hat{T}A\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^3\hat{T}^{-1}$$

and etc.

Definition 3.9. *Let $A \in \mathcal{L}(H)$ and there exists A^{-1}. The corresponding lift*

$$\hat{A}^{-1} = A^{-1}\hat{T}^{-1}$$

will be called iso-inverse iso-operator of the iso-operator A.

From here follows that, using the definition of composition of two iso-operators,

$$\hat{A}^{-1}\hat{A} = A^{-1}\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^{-1}A\hat{T}^{-1} = \hat{T}^{-1} = \hat{I},$$

$$\hat{A}\hat{A}^{-1} = A\hat{T}^{-1}\hat{T}A^{-1}\hat{T}^{-1} = AA^{-1}\hat{T}^{-1} = \hat{T}^{-1} = \hat{I}.$$

Remark 3.10. *If we define $\hat{A}^{-1}$ as follows*

$$\hat{A}^{-1} = \left(A\hat{T}^{-1}\right)^{-1}$$

then

$$\hat{A}^{-1} = \hat{T}A^{-1}.$$

From here we obtain

$$\hat{A}\hat{A}^{-1} = \hat{I} \iff$$

$$A\hat{T}^{-1}\hat{T}\hat{T}A^{-1} = \hat{T}^{-1} \iff$$

$$A\hat{T}A^{-1} = \hat{T}^{-1} \iff$$

$$\hat{T}A\hat{T}A^{-1} = I \iff$$

$$\hat{T}A\hat{T} = A$$

and

$$\hat{A}^{-1}\hat{A} = \hat{I} \iff$$

$$\hat{T}A^{-1}\hat{T}A\hat{T}^{-1} = \hat{T}^{-1} \iff$$

$$\hat{T}A^{-1}\hat{T}A = I \iff$$

$$A^{-1}\hat{T}A = \hat{T}^{-1} \iff$$

$$\hat{T}A = A\hat{T}^{-1} \iff$$

$$\hat{T}A\hat{T} = A.$$

Therefore to be built a conception for iso-inverse iso-operator we have to have the following condition between A and $\hat{T}$:

$$\hat{T}A\hat{T} = A.$$

This conception will be considered in the second part of this article. Here we will work with the above definition.

Theorem 3.11. *Let A, $B \in \mathcal{L}(H)$ and there exist A^{-1}, B^{-1}. Then*

$$\hat{A}^{-1}\hat{B}^{-1} = \widehat{BA}^{-1}.$$

Proof.

$$\hat{A}^{-1}\hat{B}^{-1} = A^{-1}\hat{T}^{-1}\hat{T}B^{-1}\hat{T}^{-1} = A^{-1}B^{-1}\hat{T}^{-1} = (BA)^{-1}\hat{T}^{-1} = \widehat{BA}^{-1}.$$

□

Definition 3.12. *Let $A \in \mathcal{L}(H)$ and A^* is its adjoint operator. Then the lift*

$$\hat{A}^* = A^*\hat{T}^{-1}$$

will be called iso-adjoint iso-operator of the iso-operator $\hat{A}$.

Here we use the word like "iso-adjoint" because in the general case we have

$$(A\hat{T}^{-1})^* \neq A^*\hat{T}^{-1}.$$

Theorem 3.13.

$$\hat{A}^*\hat{B}^* = \widehat{BA}^*.$$

Proof.

$$\hat{A}^*\hat{B}^* = A^*\hat{T}^{-1}\hat{T}B^*\hat{T}^{-1}$$

$$= A^*B^*\hat{T}^{-1}$$

$$= (BA)^*\hat{T}^{-1}$$

$$= \widehat{BA}^*.$$

□

Definition 3.14. *Let $A \in \mathcal{L}(H)$ be operator of orthogonal projection. Then the lift*

$$\hat{A} = A\hat{T}^{-1}$$

will be called iso-operator of iso-orthogonal projection.

Theorem 3.15. *Let $\hat{A}$ be iso-operator of iso-orthogonal projection. Then*

$$1) \quad \hat{A}^2 = \hat{A},$$

$$2) \quad \hat{A}^* = \hat{A}.$$

Proof.

$$1) \quad \hat{A}^2 = \hat{A}\hat{A} = A\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^2\hat{T}^{-1} = A\hat{T}^{-1} = \hat{A},$$

$$2) \quad \hat{A}^* = A^*\hat{T}^{-1} = A\hat{T}^{-1} = \hat{A}.$$

In the last representations we use the definition for iso-operator of iso-orthogonal projection and from it $A^* = A$ and $A^2 = A$. □

Let $||\cdot||$ is the norm determined by the inner product $(\cdot,\cdot)$ in H. The lift of this norm is

$$\widehat{||\hat{\cdot}||} = ||\cdot||\frac{1}{\hat{T}_1}.$$

For $\hat{x} \in \hat{H}$ we have

$$\begin{aligned}\widehat{||\hat{x}||}\hat{\times}\widehat{||\hat{x}||} &= ||\hat{T}^{-1}x||\tfrac{1}{\hat{T}_1}\hat{T}_1||||\hat{T}^{-1}x||\tfrac{1}{\hat{T}_1} \\ &= ||\hat{T}^{-1}x||^2\tfrac{1}{\hat{T}_1} \qquad (3.2) \\ &= (\hat{T}^{-1}x, \hat{T}^{-1}x)\tfrac{1}{\hat{T}_1},\end{aligned}$$

and

$$\widehat{(\hat{x},\hat{x})} = (\hat{T}^{-1}x, \hat{T}^{-1}x)\frac{1}{\hat{T}_1}.$$

From here and (3.2) follows that

$$\widehat{||\hat{x}||}\hat{\times}\widehat{||\hat{x}||} = \widehat{(\hat{x},\hat{x})}.$$

Definition 3.16. *Let $\{\hat{T}^n\}_{n=1}^{\infty}$ satisfy $(H1)$. We will say that the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ of iso-elements of $\hat{\mathcal{L}}(\hat{H})$ is uniformly convergent to $\hat{A} \in \hat{\mathcal{L}}(\hat{H})$ if*

$$\widehat{||\hat{A}_n - \hat{A}||} \longrightarrow_{n\longrightarrow\infty} 0.$$

Remark 3.17. *We will note that from the uniformly convergence of $\{\hat{A}_n\}_{n=1}^{\infty}$ to $\hat{A}$ does not foolow the uniformly convergence of $\{A_n\}_{n=1}^{\infty}$ to A and the inverse. We will see this with some examples.*

Examples 3.18. *Let* $A, B, \hat{T}^{-1}, A_n, \hat{T}^{-1n} : H \longrightarrow H$ *and for* $x \in H$

$$A_n x = \tfrac{n^2+1}{2n^2+3}x, \quad \hat{T}^{-1n}x = \tfrac{n+1}{2n+1}x,$$

$$\hat{T}^{-1}x = \tfrac{1}{4}x, \quad Ax = x, \quad Bx = \tfrac{1}{2}x.$$

Then, for $x \in H$,

$$||A_n x - Ax|| = \left|\left|\tfrac{n^2+1}{2n^2+3}x - x\right|\right| = \left|\left|\left(\tfrac{n^2+1}{2n^2+3} - 1\right)x\right|\right|,$$

$$||A_n - A|| = \sup_{x\in H, ||x||\leq 1}\left|1 - \tfrac{n^2+1}{2n^2+3}\right| ||x||$$

$$= \left|1 - \tfrac{n^2+1}{2n^2+3}\right|$$

from here

$$\lim_{n\longrightarrow\infty} ||A_n - A|| = \lim_{n\longrightarrow\infty}\left|1 - \frac{n^2+1}{2n^2+3}\right| = 1 - \frac{1}{2} = \frac{1}{2},$$

consequently the sequence $\{A_n\}_{n=1}^{\infty}$ *does not converge uniformly to* A.

But, for $\hat{x} \in \hat{H}$,

$$\hat{A}_n\hat{x} = A_n\hat{T}^{-1n}x = A_n\left(\tfrac{n+1}{2n+3}x\right) = \tfrac{(n^2+1)(n+1)}{(2n^2+3)(2n+1)}x,$$

$$\hat{A}\hat{x} = A\hat{T}^{-1}x = A\left(\tfrac{1}{4}x\right) = \tfrac{1}{4}x,$$

from where

$$||\widehat{\hat{A}_n - \hat{A}}|| = ||A_n\hat{T}^{-1n}x - A\hat{T}^{-1}x||_{\frac{1}{T_1}}$$

$$= \left|\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)}x - \tfrac{1}{4}x\right|\right|_{\frac{1}{T_1}}$$

$$= \left|\left|\left(\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right)x\right|\right|_{\frac{1}{T_1}}$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right|||x||_{\frac{1}{T_1}},$$

$$||\widehat{\hat{A}_n - \hat{A}}|| = ||A_n\hat{T}^{-1n} - A\hat{T}^{-1}||$$

$$= \sup_{x\in H,||x||\leq 1}\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right|||x||_{\frac{1}{T_1}}$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right|_{\frac{1}{T_1}},$$

and then

$$\lim_{n\longrightarrow\infty}||\widehat{\hat{A}_n - \hat{A}}|| = \lim_{n\longrightarrow\infty}\left|\frac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \frac{1}{4}\right|\frac{1}{T_1} = 0,$$

consequently the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ *is uniformly convergent to* $\hat{A}$.

Also, for $\hat{x} \in \hat{H}$,

$$\hat{B}\hat{x} = B\hat{T}^{-1}x = B\left(\tfrac{1}{4}x\right) = \tfrac{1}{8}x,$$

from where

$$\widehat{||\hat{A}_n - \hat{B}||} = ||A_n\hat{T}^{-1n}x - B\hat{T}^{-1}x||$$

$$= \left|\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)}x - \tfrac{1}{8}x\right|\right|\tfrac{1}{T_1}$$

$$= \left|\left|\left(\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right)x\right|\right|\tfrac{1}{T_1}$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right|\tfrac{1}{T_1}||x||,$$

$$\widehat{||\hat{A}_n - \hat{B}||} = ||A_n\hat{T}^{-1n} - B\hat{T}^{-1}||\tfrac{1}{T_1}$$

$$= \sup_{x\in H,||x||\leq 1}\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right|\tfrac{1}{T_1}||x||$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right|\tfrac{1}{T_1},$$

and then

$$\lim_{n\longrightarrow\infty}\widehat{||\hat{A}_n - \hat{A}||} = \lim_{n\longrightarrow\infty}\left|\frac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \frac{1}{8}\right|\frac{1}{T_1} \neq 0,$$

consequently the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ *is not uniformly convergent to* $\hat{B}$.

But, for $x \in H$,

$$||A_nx - Bx|| = \left|\left|\tfrac{n^2+1}{2n^2+3}x - \tfrac{1}{2}x\right|\right| = \left|\left|\left(\tfrac{n^2+1}{2n^2+3} - \tfrac{1}{2}\right)x\right|\right|,$$

$$||A_n - B|| = \sup_{x\in H,||x||\leq 1}\left|\tfrac{1}{2} - \tfrac{n^2+1}{2n^2+3}\right||x||$$

$$= \left|\tfrac{1}{2} - \tfrac{n^2+1}{2n^2+3}\right|$$

from here

$$\lim_{n\longrightarrow\infty}||A_n - B|| = \lim_{n\longrightarrow\infty}\left|\frac{1}{2} - \frac{n^2+1}{2n^2+3}\right| = 0,$$

consequently the sequence $\{A_n\}_{n=1}^{\infty}$ *is uniformly convergent to* B.

Definition 3.19. *Let $\{\hat{T}^n\}_{n=1}^{\infty}$ satisfy $(H1)$. We will say that the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ of iso-elements of $\hat{\mathcal{L}}(\hat{H})$ is strongly convergent to $\hat{A} \in \hat{\mathcal{L}}(\hat{H})$ if*

$$\widehat{||\hat{A}_n\hat{x} - \hat{A}\hat{x}||} \longrightarrow_{n\longrightarrow\infty} 0$$

for every $\hat{x} \in \hat{H}$.

Remark 3.20. *From the above examples follows that from the strongly convergence of $\{\hat{A}_n\}_{n=1}^{\infty}$ to $\hat{A}$ does not follow the strongly convergence of $\{A_n\}_{n=1}^{\infty}$ to A and the inverse.*

Theorem 3.21. *Let $\{\hat{T}^n\}_{n=1}^{\infty}$ satisfy $(H1)$ and $\{\hat{A}_n\}_{n=1}^{\infty}$ is uniformly convergent to $\hat{A} \in \hat{\mathcal{L}}(\hat{H})$. Then $\{\hat{A}_n\}_{n=1}^{\infty}$ is strongly convergent to $\hat{A} \in \hat{\mathcal{L}}(\hat{H})$.*

Proof. The proof follows from the following iso-inequality

$$\widehat{||\hat{A}_n\hat{x} - \hat{A}\hat{x}||} \leq \widehat{||\hat{A}_n - \hat{A}||}\hat{\times}\widehat{||\hat{x}||}.$$

□

Definition 3.22. *Every linear iso-operator $\hat{L} : \hat{H} \longrightarrow \hat{F}_{\mathbb{R}}$ will be called linear iso-functional.*

Definition 3.23. *The iso-sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ of iso-elements of $\hat{H}$ will be called weakly convergent to $\hat{x} \in \hat{H}$ if*

$$\hat{L}(\hat{x}_n) \longrightarrow_{n\longrightarrow\infty} \hat{L}(\hat{x})$$

for every linear iso-functional $\hat{L}$ defined on $\hat{H}$.

If $\hat{L}$ is linear iso-functional on $\hat{H}$ then for every $\hat{x} \in \hat{H}$ we have

$$\widehat{|\hat{L}(\hat{x})|} \leq \widehat{||\hat{L}||}\hat{\times}\widehat{||\hat{x}||}.$$

Theorem 3.24. *Let $\{\hat{x}_n\}_{n=1}^{\infty}$ be a sequence of iso-elements of $\hat{H}$ which is strongly convergent to $\hat{x} \in \hat{H}$. Then it is weakly convergent.*

Proof. Let $\hat{L}$ be arbitrary linear iso-functional on $\hat{H}$. Then we have

$$|\hat{L}(\hat{x}_n) - \hat{L}(\hat{x})| = |\hat{L}(\hat{x}_n - \hat{x})| \leq \widehat{||\hat{L}||}\widehat{||\hat{x}_n - \hat{x}||}. \tag{3.3}$$

Since $\{\hat{x}_n\}_{n=1}^{\infty}$ is strongly convergent to $\hat{x}$ we have

$$\lim_{n\longrightarrow\infty} \widehat{||\hat{x}_n - \hat{x}||} = 0,$$

from here and (3.3) we conclude

$$\lim_{n\longrightarrow\infty} |\hat{L}(\hat{x}_n) - \hat{L}(\hat{x})| = 0,$$

because $\hat{L}$ was arbitrary chosen then $\{\hat{x}_n\}_{n=1}^{\infty}$ is weakly convergent to $\hat{x}$. □

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THE ISO-DUAL TESSERACT

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Abstract

In this work, we deploy Santilli's iso-dual iso-topic lifting and Inopin's holographic ring (IHR) topology as a platform to introduce and assemble a tesseract from two inter-locking, iso-morphic, iso-dual cubes in Euclidean triplex space. For this, we prove that such an "iso-dual tesseract" can be constructed by following a procedure of simple, flexible, topologically-preserving instructions. Moreover, these novel results are significant because the tesseract's state and structure are directly inferred from the one initial cube (rather than two distinct cubes), which identifies a new iso-geometrical inter-connection between Santilli's exterior and interior dynamical systems.

Keywords: Santilli iso-number; Inopin holographic ring; Iso-geometry; Tesseract; Topology.

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1 Introduction

Everybody knows what the square is: a square is a 2D object in 2D space with 4 equal edges, 4 equal angles, and 4 vertices. Most know what the cube is: a cube is a 3D object in 3D space—the 3D analog of the square—with 12 equal edges, 6 square faces, and 8 vertices, where 3 edges meet at each vertex. But few know what the *tesseract* is: a tesseract is a 4D object in 4D space—the 4D analog of the cube—with 32 edges, 24 faces, and 16 vertices, where 4 edges meet at each vertex. Basically, the tesseract is to the cube just as the cube is to the square.

To date, there are numerous *geometrical* procedures of tesseract construction that operate with conventional *mathematics*. However, in this paper, we disclose the first *iso-geometrical* procedure of tesseract construction that operates with Santilli's new *iso-mathematics* [1, 2, 3, 4, 5, 6].

To introduce and illustrate this notion, lets consider one approach to build a tesseract. First, we know that the cube has 8 vertices and the tesseract has 16 vertices, therefore a tesseract has *two* times as many vertices as a cube. For this method, this value of *two* is of interest to us—but why? Well, suppose that *two* distinct cubes are positioned in a 3D space, where the sum of the vertices of these two cubes is 16. These resulting 16 vertices indicate that a tesseract can be assembled from the two cubes by introducing 8 additional edges to inter-connect the 8 vertex pairs in a pairwise fashion. Now lets take this one step further: what if one could build a tesseract from a *one* cube instead of two? In conventional mathematics, this question may seem irrelevant because the 8 vertices of a single cube is insufficient to synthesize a tesseract of 16 vertices. However, in the realm of Santilli's *iso-mathematics* [1, 2, 3, 4, 5, 6], this question becomes legitimate when we consider the concept of *iso-duality*.

In this paper, we attack the said inquiry and prove that it is possible to build a tesseract from one *initial cube* by iso-topically lifting [1, 2, 3, 4, 5, 6] its 8 vertices to simultaneously infer an *exterior cube* and an *interior cube* to generate the required 16 vertices, where the double-projected cubes are iso-dual and are both iso-morphic, inter-locking, and synchronized to the initial cube. Consequently, the 16 generated vertices are inter-connected in a pairwise fashion to iso-mathematically synthesize the *iso-tesseract*. Thus, for this investigation, Section 2 presents the main procedure and results,

while Section 3 recapitulates the significance of our discovery and suggests future modes of exploration along this research trajectory.

2 Procedure

In this main section, we launch our exploration by instantiating the dual 4D space-time IHR topology [6, 7, 8, 9] so we can subsequently generalize it to encompass the exterior and interior IHR iso-duality [10] and thereby assemble the iso-dual tesseract from one cube through a step-by-step process.

2.1 Preparation: initializing the dual 4D space-time IHR topology

Here, we prepare for the iso-dual tesseract construction of Section 2.2 by first recalling the dual 4D space-time IHR topology [6, 7, 8, 9, 10] via the following procedure:

1. First, given eq. (18) of [6] we identify $Y \equiv \mathbb{T}$ as the set of all triplex numbers, the Euclidean triplex space, and the *dual 3D Cartesian-spherical coordinate-vector state space*. Here, a triplex number $\vec{y} \in Y$ is a *dual 3D Cartesian-spherical coordinate-vector state* that is expressed via eq. (17) of [6] as

$$y = \vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_{Z} = (\vec{y}) = (|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S = (\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C, \quad \forall \vec{y} \in Y, \tag{1}$$

   where $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C$ is a *3D Cartesian coordinate-vector state* in the *3D Cartesian coordinate-vector state space* Y_C so $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C \in Y_C$, while simultaneously $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S$ is a *3D spherical coordinate-vector state* in the *3D spherical coordinate-vector state space* Y_S so $(|\vec{y}|, \langle\vec{y}\rangle, [\vec{y}])_S \in Y_S$, such that Y_C and Y_S are dual, iso-morphic, synchronized, and interlocking in Y [6]. Hence, eq. (1) satisfies with the constraints imposed by eqs. (19–28) of [6]—see Figures 4 and 5 of [6].

2. Second, given eq. (33) of [6] we recall that

$$T_r^2 = \{\vec{y} \in Y : |\vec{y}| = r\}, \tag{2}$$

   where $T_r^2 \subset Y$ is the *2-sphere IHR* of amplitude-radius $r > 0$ that is centered on the origin $O \in Y$ and is iso-metrically embedded in Y; T_r^2

is the multiplicative group of all non-zero triplex numbers with the amplitude-radius r, which is simultaneously dual to the two triplex sub-spaces [6, 7, 8]: the "micro sub-space 3-brane" $Y_- \subset Y$ and the "macro sub-space 3-brane" $Y_+ \subset Y$—see Figure 7 of [6]. Here, we note that the *1-sphere IHR* $T^1_r \subset T^2_r$ of amplitude-radius $r > 0$ (and curvature $\frac{1}{r}$) from eq. (16) of [6] is the great circle of T^2_r.

At this point, we've initialized Inopin's dual 4D space-time IHR topology [6, 7, 8, 9, 10] and are therefore prepared to assemble the iso-dual tesseract of Section 2.2.

2.2 Engagement: constructing the iso-dual tesseract

Here, equipped with the dual 4D space-time IHR topology of Section 2.1, we introduce, define, and assemble the proposed iso-dual tesseract via the following procedure:

1. First, we recall that in conventional mathematics the number 1 is the multiplicative identity which satisfies the original number field axioms [11]. Thus, in general, the number 1 plays a crucial and diverse role throughout the various branches of mathematics such as, for example, normalization in statistics. Therefore, we begin by setting the amplitude-radius $r = 1$ for T^1_r and T^2_r.

2. Second, we construct the *initial cube* from 8 triplex vertices that are confined to T^2_r. For this cube, we define the underlying set of 8 triplex vertices as

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T^2_r} \subset T^2_r \subset Y \tag{3}$$

such that

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r\} \equiv V^{\uparrow}_{T^2_r} \subset V_{T^2_r} \subset T^2_r \subset Y \tag{4}$$

are the "top vertices" for the "top square surface" and

$$\{\bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V^{\downarrow}_{T^2_r} \subset V_{T^2_r} \subset T^2_r \subset Y \tag{5}$$

are the "bottom vertices" for the "bottom square surface", which comply with the *cubic vertex triplex amplitude-radius constraints*

$$\begin{aligned} 1 \equiv r \equiv |\vec{a}_r| \equiv |\vec{b}_r| \equiv |\vec{c}_r| \equiv |\vec{d}_r| \\ \equiv |\bar{\vec{a}}_r| \equiv |\bar{\vec{b}}_r| \equiv |\bar{\vec{c}}_r| \equiv |\bar{\vec{d}}_r|, \end{aligned} \tag{6}$$

the *cubic vertex triplex phase constraints*

$$\begin{aligned} \langle\vec{a}_r\rangle &\equiv \langle\vec{b}_r\rangle - \tfrac{\pi}{2} \equiv \langle\vec{c}_r\rangle \pm \pi \equiv \langle\vec{d}_r\rangle - \tfrac{3\pi}{2} \\ \langle\bar{\vec{a}}_r\rangle &\equiv \langle\bar{\vec{b}}_r\rangle - \tfrac{\pi}{2} \equiv \langle\bar{\vec{c}}_r\rangle \pm \pi \equiv \langle\bar{\vec{d}}_r\rangle - \tfrac{3\pi}{2} \end{aligned} \tag{7}$$

such that

$$\begin{aligned} \langle\vec{a}_r\rangle &\equiv \langle\bar{\vec{a}}_r\rangle \pm \pi \\ \langle\vec{b}_r\rangle &\equiv \langle\bar{\vec{b}}_r\rangle \pm \pi \\ \langle\vec{c}_r\rangle &\equiv \langle\bar{\vec{c}}_r\rangle \pm \pi \\ \langle\vec{d}_r\rangle &\equiv \langle\bar{\vec{d}}_r\rangle \pm \pi, \end{aligned} \tag{8}$$

and the *cubic vertex triplex inclination constraints*

$$\begin{aligned} [\vec{a}_r] \equiv [\vec{b}_r] \equiv [\vec{c}_r] \equiv [\vec{d}_r] \\ [\bar{\vec{a}}_r] \equiv [\bar{\vec{b}}_r] \equiv [\bar{\vec{c}}_r] \equiv [\bar{\vec{d}}_r] \end{aligned} \tag{9}$$

such that

$$\begin{aligned} [\vec{a}_r] &\equiv [\bar{\vec{a}}_r] \pm \pi \\ [\vec{b}_r] &\equiv [\bar{\vec{b}}_r] \pm \pi \\ [\vec{c}_r] &\equiv [\bar{\vec{c}}_r] \pm \pi \\ [\vec{d}_r] &\equiv [\bar{\vec{d}}_r] \pm \pi, \end{aligned} \tag{10}$$

to establish the *cubic vertex triplex antisymmetric constraints*

$$\begin{aligned} \vec{a}_r &\equiv -\bar{\vec{a}}_r \\ \vec{b}_r &\equiv -\bar{\vec{b}}_r \\ \vec{c}_r &\equiv -\bar{\vec{c}}_r \\ \vec{d}_r &\equiv -\bar{\vec{d}}_r. \end{aligned} \tag{11}$$

Therefore, the cube built from the 8 triplex vertices comprising $V_{T_r^2}$ of eq. (3)—which satisfy eqs. (6–11) and are confined to T_r^2—is depicted in Figures 1 and 2.

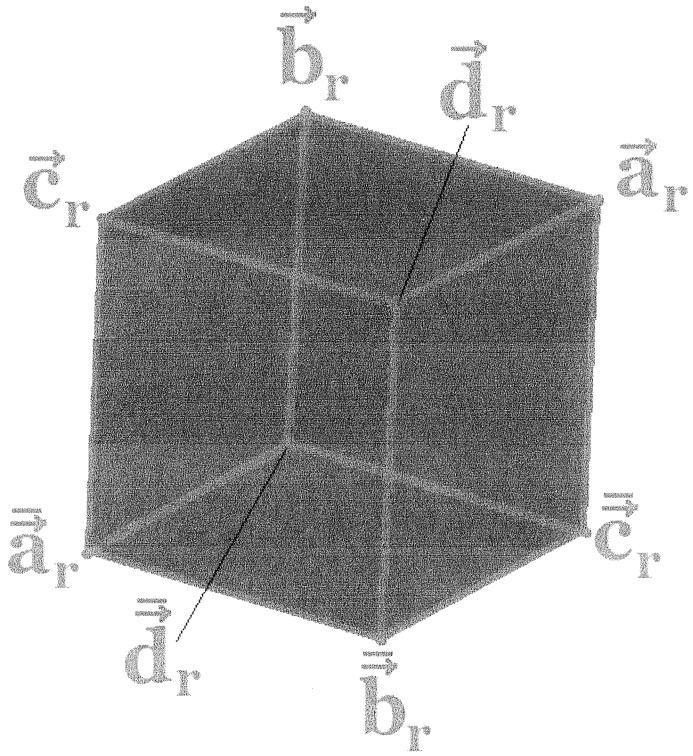

Fig. 1: The 8 triplex vertices of the initial cube comprise the set $\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T_r^2}$, which are confined to T_r^2 (not shown).

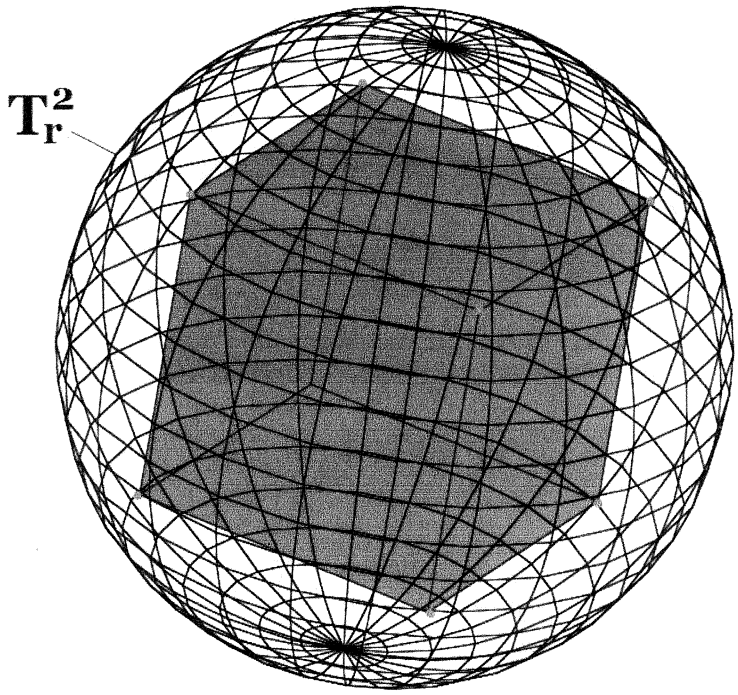

Fig. 2: The 8 triplex vertices of the initial cube comprise the set $\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r\} \equiv V_{T_r^2}$, which are confined to T_r^2 (shown).

3. Third, in iso-mathematics [1, 2, 3, 4, 5], Santilli proved that the standard multiplicative unit which satisfies the conventional number field axioms [11] is not limited to the number 1, and can thus be replaced with a positive-definite iso-multiplicative iso-unit $\hat{r}_+ > 0$ for iso-numbers. Thus, in accordance to Santilli's methodology [1, 2, 3, 4, 5], we select some $\hat{r}_+$ with the corresponding iso-unit inverse $\hat{r}_- = \frac{1}{\hat{r}_+}$, such that
$$\hat{r}_+ > r > \hat{r}_- > 0. \tag{12}$$

4. Fourth, we engage $\hat{r}_+$ to iso-topically lift [1, 2, 3, 4, 5] T_r^2 to the *exterior iso-2-sphere IHR* $T_{\hat{r}_+}^2$ via the transition
$$f(T_r^2, \hat{r}_+): \quad T_r^2 \to T_{\hat{r}_+}^2 \tag{13}$$
and its corresponding inverse
$$f^{-1}(T_{\hat{r}_+}^2, \hat{r}_+): \quad T_{\hat{r}_+}^2 \to T_r^2, \tag{14}$$
such that the iso-unit $\hat{r}_+$ is the *exterior iso-radius* of $T_{\hat{r}_+}^2$, which is "outside" of T_r^2 because eq. (2) becomes
$$T_{\hat{r}_+}^2 \equiv \{\vec{y}_{\hat{r}_+} \in Y : |\vec{y}_{\hat{r}_+}| = r \times \hat{r}_+\} \tag{15}$$
for
$$\vec{y}_{\hat{r}_+} \equiv \vec{y} \times \hat{r}_+, \ \forall \vec{y} \in T_r^2 \to \forall \vec{y}_{\hat{r}_+} \in T_{\hat{r}_+}^2, \tag{16}$$
where T_r^2 and $T_{\hat{r}_+}^2$ are *locally iso-morphic* [6, 10]. Therefore, given that $V_{T_r^2} \subset T_r^2$, the iso-topic lifting of eqs. (13–16) indicates
$$\begin{array}{rcl} \vec{a}_{\hat{r}_+} & \equiv & \vec{a}_r \times \hat{r}_+ \\ \vec{b}_{\hat{r}_+} & \equiv & \vec{b}_r \times \hat{r}_+ \\ \vec{c}_{\hat{r}_+} & \equiv & \vec{c}_r \times \hat{r}_+ \\ \vec{d}_{\hat{r}_+} & \equiv & \vec{d}_r \times \hat{r}_+ \\ \vec{\bar{a}}_{\hat{r}_+} & \equiv & \vec{\bar{a}}_r \times \hat{r}_+ \\ \vec{\bar{b}}_{\hat{r}_+} & \equiv & \vec{\bar{b}}_r \times \hat{r}_+ \\ \vec{\bar{c}}_{\hat{r}_+} & \equiv & \vec{\bar{c}}_r \times \hat{r}_+ \\ \vec{\bar{d}}_{\hat{r}_+} & \equiv & \vec{\bar{d}}_r \times \hat{r}_+, \end{array} \tag{17}$$

enabling us to rewrite eq. (6) to establish the *exterior cubic iso-vertex iso-triplex amplitude-radius constraints*

$$\begin{array}{rcccccccc} \hat{r}_+ & \equiv & |\vec{a}_{\hat{r}_+}| & \equiv & |\vec{b}_{\hat{r}_+}| & \equiv & |\vec{c}_{\hat{r}_+}| & \equiv & |\vec{d}_{\hat{r}_+}| \\ & \equiv & |\bar{\vec{a}}_{\hat{r}_+}| & \equiv & |\bar{\vec{b}}_{\hat{r}_+}| & \equiv & |\bar{\vec{c}}_{\hat{r}_+}| & \equiv & |\bar{\vec{d}}_{\hat{r}_+}| \end{array} \tag{18}$$

with the *exterior iso-vertex* directional-preservations

$$\begin{array}{rcccccc} \langle \vec{a}_{\hat{r}_+} \rangle & \equiv & \langle \vec{a}_r \rangle & | & [\vec{a}_{\hat{r}_+}] & \equiv & [\vec{a}_r] \\ \langle \vec{b}_{\hat{r}_+} \rangle & \equiv & \langle \vec{b}_r \rangle & | & [\vec{b}_{\hat{r}_+}] & \equiv & [\vec{b}_r] \\ \langle \vec{c}_{\hat{r}_+} \rangle & \equiv & \langle \vec{c}_r \rangle & | & [\vec{c}_{\hat{r}_+}] & \equiv & [\vec{c}_r] \\ \langle \vec{d}_{\hat{r}_+} \rangle & \equiv & \langle \vec{d}_r \rangle & | & [\vec{d}_{\hat{r}_+}] & \equiv & [\vec{d}_r] \\ \langle \bar{\vec{a}}_{\hat{r}_+} \rangle & \equiv & \langle \bar{\vec{a}}_r \rangle & | & [\bar{\vec{a}}_{\hat{r}_+}] & \equiv & [\bar{\vec{a}}_r] \\ \langle \bar{\vec{b}}_{\hat{r}_+} \rangle & \equiv & \langle \bar{\vec{b}}_r \rangle & | & [\bar{\vec{b}}_{\hat{r}_+}] & \equiv & [\bar{\vec{b}}_r] \\ \langle \bar{\vec{c}}_{\hat{r}_+} \rangle & \equiv & \langle \bar{\vec{c}}_r \rangle & | & [\bar{\vec{c}}_{\hat{r}_+}] & \equiv & [\bar{\vec{c}}_r] \\ \langle \bar{\vec{d}}_{\hat{r}_+} \rangle & \equiv & \langle \bar{\vec{d}}_r \rangle & | & [\bar{\vec{d}}_{\hat{r}_+}] & \equiv & [\bar{\vec{d}}_r] \end{array} \tag{19}$$

that continue to satisfy the generalized constraints of eqs. (7–11) to establish

$$\{\vec{a}_{\hat{r}_+}, \vec{b}_{\hat{r}_+}, \vec{c}_{\hat{r}_+}, \vec{d}_{\hat{r}_+}, \bar{\vec{a}}_{\hat{r}_+}, \bar{\vec{b}}_{\hat{r}_+}, \bar{\vec{c}}_{\hat{r}_+}, \bar{\vec{d}}_{\hat{r}_+}\} \equiv V_{T^2_{\hat{r}_+}} \subset T^2_{\hat{r}_+} \subset Y_+ \tag{20}$$

for the implied exterior vertex iso-topic lifting $V_{T^2_r} \rightarrow V_{T^2_{\hat{r}_+}}$, where $V_{T^2_{\hat{r}_+}}$ is the exterior set of 8 iso-triplex iso-vertices that are confined to $T^2_{\hat{r}_+}$ and form the *exterior cube* of the tesseract for the *exterior dynamical system* of the macro sub-space 3-brane Y_+.

5. Fifth, given eqs. (13–16), the relation $\hat{r}_- = \frac{1}{\hat{r}_+}$ is the foundation of the exterior and interior IHR iso-duality of [10], where the iso-unit inverse $\hat{r}_-$ is the *interior iso-radius* of the *interior iso-2-sphere IHR* $T^2_{\hat{r}_-}$ that is "inside" of T^2_r, such that T^2_r is simultaneously iso-topically lifted to $T^2_{\hat{r}_-}$ via the transition

$$f(T^2_r, \hat{r}_-): \quad T^2_r \rightarrow T^2_{\hat{r}_-} \tag{21}$$

and its corresponding inverse

$$f^{-1}(T^2_{\hat{r}_-}, \hat{r}_-): \quad T^2_{\hat{r}_-} \rightarrow T^2_r, \tag{22}$$

because eq. (2) becomes

$$T^2_{\hat{r}_-} \equiv \{\hat{\vec{y}} \in Y : |\hat{\vec{y}}| = r \times \hat{r}_-\} \tag{23}$$

for

$$\hat{\vec{y}} \equiv \vec{y} \times \hat{r}_-, \ \forall \vec{y} \in T^2_r \rightarrow \forall \hat{\vec{y}} \in T^2_{\hat{r}_-}, \tag{24}$$

where T^2_r and $T^2_{\hat{r}_-}$ are *locally iso-morphic* [6, 10]. Thus, the $T^2_{\hat{r}_+}$ of eqs. (13–16) is *iso-dual* to the $T^2_{\hat{r}_-}$ of eqs. (21–24) with respect to T^2_r in accordance to the exterior and interior IHR iso-duality of [10]. Therefore, given that $V_{T^2_r} \subset T^2_r$, the iso-topic lifting of eqs. (21–24) indicates

$$\begin{array}{rcl}
\vec{a}_{\hat{r}_-} & \equiv & \vec{a}_r \times \hat{r}_- \\
\vec{b}_{\hat{r}_-} & \equiv & \vec{b}_r \times \hat{r}_- \\
\vec{c}_{\hat{r}_-} & \equiv & \vec{c}_r \times \hat{r}_- \\
\vec{d}_{\hat{r}_-} & \equiv & \vec{d}_r \times \hat{r}_- \\
\bar{\vec{a}}_{\hat{r}_-} & \equiv & \bar{\vec{a}}_r \times \hat{r}_- \\
\bar{\vec{b}}_{\hat{r}_-} & \equiv & \bar{\vec{b}}_r \times \hat{r}_- \\
\bar{\vec{c}}_{\hat{r}_-} & \equiv & \bar{\vec{c}}_r \times \hat{r}_- \\
\bar{\vec{d}}_{\hat{r}_-} & \equiv & \bar{\vec{d}}_r \times \hat{r}_-,
\end{array} \tag{25}$$

enabling us to rewrite eq. (6) to establish the *interior cubic iso-vertex iso-triplex amplitude-radius constraints*

$$\begin{array}{rcccccccc}
\hat{r}_- & \equiv & |\vec{a}_{\hat{r}_-}| & \equiv & |\vec{b}_{\hat{r}_-}| & \equiv & |\vec{c}_{\hat{r}_-}| & \equiv & |\vec{d}_{\hat{r}_-}| \\
& \equiv & |\bar{\vec{a}}_{\hat{r}_-}| & \equiv & |\bar{\vec{b}}_{\hat{r}_-}| & \equiv & |\bar{\vec{c}}_{\hat{r}_-}| & \equiv & |\bar{\vec{d}}_{\hat{r}_-}|
\end{array} \tag{26}$$

with the *interior iso-vertex* directional-preservations

$$\begin{array}{ccccccc|ccccc}
\langle \vec{a}_{\hat{r}_-} \rangle & \equiv & \langle \vec{a}_r \rangle & \equiv & \langle \vec{a}_{\hat{r}_+} \rangle & | & [\vec{a}_{\hat{r}_-}] & \equiv & [\vec{a}_r] & \equiv & [\vec{a}_{\hat{r}_+}] \\
\langle \vec{b}_{\hat{r}_-} \rangle & \equiv & \langle \vec{b}_r \rangle & \equiv & \langle \vec{b}_{\hat{r}_+} \rangle & | & [\vec{b}_{\hat{r}_-}] & \equiv & [\vec{b}_r] & \equiv & [\vec{b}_{\hat{r}_+}] \\
\langle \vec{c}_{\hat{r}_-} \rangle & \equiv & \langle \vec{c}_r \rangle & \equiv & \langle \vec{c}_{\hat{r}_+} \rangle & | & [\vec{c}_{\hat{r}_-}] & \equiv & [\vec{c}_r] & \equiv & [\vec{c}_{\hat{r}_+}] \\
\langle \vec{d}_{\hat{r}_-} \rangle & \equiv & \langle \vec{d}_r \rangle & \equiv & \langle \vec{d}_{\hat{r}_+} \rangle & | & [\vec{d}_{\hat{r}_-}] & \equiv & [\vec{d}_r] & \equiv & [\vec{d}_{\hat{r}_+}] \\
\langle \bar{\vec{a}}_{\hat{r}_-} \rangle & \equiv & \langle \bar{\vec{a}}_r \rangle & \equiv & \langle \bar{\vec{a}}_{\hat{r}_+} \rangle & | & [\bar{\vec{a}}_{\hat{r}_-}] & \equiv & [\bar{\vec{a}}_r] & \equiv & [\bar{\vec{a}}_{\hat{r}_+}] \\
\langle \bar{\vec{b}}_{\hat{r}_-} \rangle & \equiv & \langle \bar{\vec{b}}_r \rangle & \equiv & \langle \bar{\vec{b}}_{\hat{r}_+} \rangle & | & [\bar{\vec{b}}_{\hat{r}_-}] & \equiv & [\bar{\vec{b}}_r] & \equiv & [\bar{\vec{b}}_{\hat{r}_+}] \\
\langle \bar{\vec{c}}_{\hat{r}_-} \rangle & \equiv & \langle \bar{\vec{c}}_r \rangle & \equiv & \langle \bar{\vec{c}}_{\hat{r}_+} \rangle & | & [\bar{\vec{c}}_{\hat{r}_-}] & \equiv & [\bar{\vec{c}}_r] & \equiv & [\bar{\vec{c}}_{\hat{r}_+}] \\
\langle \bar{\vec{d}}_{\hat{r}_-} \rangle & \equiv & \langle \bar{\vec{d}}_r \rangle & \equiv & \langle \bar{\vec{d}}_{\hat{r}_+} \rangle & | & [\bar{\vec{d}}_{\hat{r}_-}] & \equiv & [\bar{\vec{d}}_r] & \equiv & [\bar{\vec{d}}_{\hat{r}_+}]
\end{array} \tag{27}$$

that incorporate eq. (19) and continue to satisfy the generalized constraints of eqs. (7–11) to establish

$$\{\vec{a}_{\hat{r}_-}, \vec{b}_{\hat{r}_-}, \vec{c}_{\hat{r}_-}, \vec{d}_{\hat{r}_-}, \bar{\vec{a}}_{\hat{r}_-}, \bar{\vec{b}}_{\hat{r}_-}, \bar{\vec{c}}_{\hat{r}_-}, \bar{\vec{d}}_{\hat{r}_-}\} \equiv V_{T^2_{\hat{r}_-}} \subset T^2_{\hat{r}_-} \subset Y_- \tag{28}$$

for the implied interior vertex iso-topic lifting $V_{T^2_r} \rightarrow V_{T^2_{\hat{r}_-}}$, where $V_{T^2_{\hat{r}_-}}$ is the interior set of 8 iso-triplex iso-vertices that are confined to $T^2_{\hat{r}_-}$ and form the *interior cube* of the tesseract for the *interior dynamical system* of the micro sub-space 3-brane Y_-.

6. Sixth, given the 8 exterior iso-triplex iso-vertices of $T^2_{\hat{r}_+}$ in eq. (20) and the 8 interior iso-triplex iso-vertices of $T^2_{\hat{r}_-}$ in eq. (28), we identify the 16 iso-triplex iso-vertices of the iso-dual tesseract as

$$V_{T^2_{\hat{r}_\pm}} \equiv V_{T^2_{\hat{r}_+}} \cup V_{T^2_{\hat{r}_-}}, \tag{29}$$

where 8 additional edges are inserted to inter-link the iso-vertex pairs in a pairwise fashion to inter-connect Santilli's exterior and interior dynamical systems for Y_+ and Y_-, respectively. See Figure 3 for a depiction of the iso-dual tesseract.

7. Seventh, it is straightforward to assign triplex order parameters [6, 7, 9] to the iso-vertices of eq. (29) to topologically deform the tesseract. For example, suppose that one layer of triplex order parameters [6, 7, 9] is assigned to the 8 vertices of $V_{T^2_r}$ as

$$\{\vec{\psi}(\vec{a}_r), \vec{\psi}(\vec{b}_r), \vec{\psi}(\vec{c}_r), \vec{\psi}(\vec{d}_r), \vec{\psi}(\bar{\vec{a}}_r), \vec{\psi}(\bar{\vec{b}}_r), \vec{\psi}(\bar{\vec{c}}_r), \vec{\psi}(\bar{\vec{d}}_r)\} \equiv \vec{\psi}_{T^2_r} \tag{30}$$

to encode topological deformations that comply with the antisymmetric constraints

$$\begin{array}{rcl} \vec{\psi}(\vec{a}_r) & \equiv & -\vec{\psi}(\bar{\vec{a}}_r) \\ \vec{\psi}(\vec{b}_r) & \equiv & -\vec{\psi}(\bar{\vec{b}}_r) \\ \vec{\psi}(\vec{c}_r) & \equiv & -\vec{\psi}(\bar{\vec{c}}_r) \\ \vec{\psi}(\vec{d}_r) & \equiv & -\vec{\psi}(\bar{\vec{d}}_r) \end{array} \tag{31}$$

that are depicted in Figure 4.

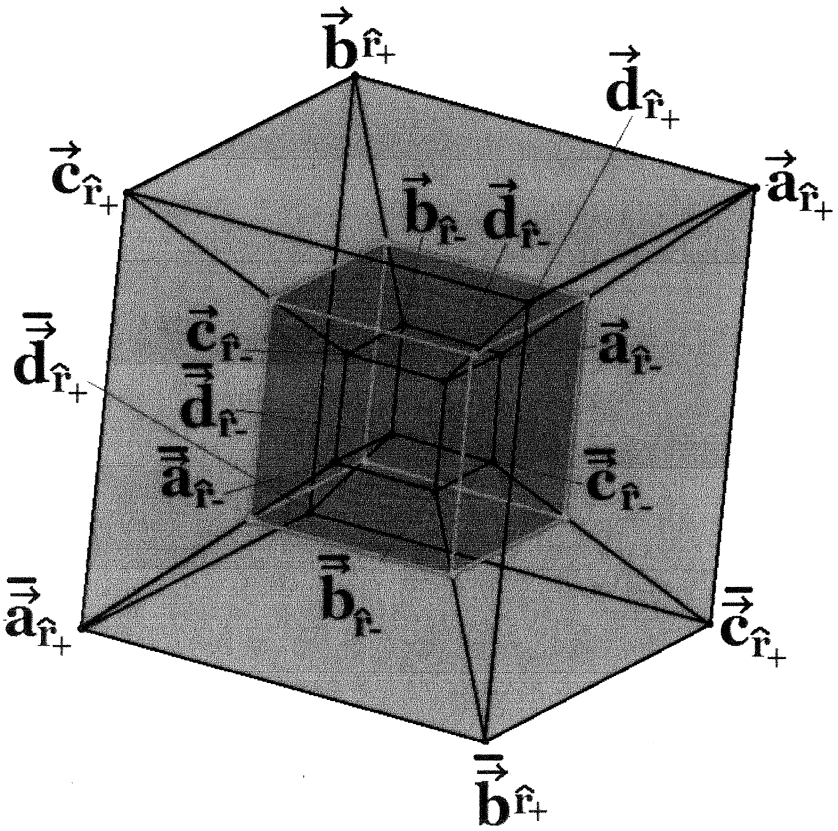

Fig. 3: The 8 triplex vertices of $V^2_{T_r} \subset T^2_r$ are iso-topically lifted via the double-projection iso-dual transition $V_{T^2_{\hat{r}_-}} \subset T^2_{\hat{r}_-} \leftarrow V_{T^2_r} \rightarrow V_{T^2_{\hat{r}_+}} \subset T^2_{\hat{r}_+}$ to generate the 16 iso-triplex iso-vertices of $V_{T^2_{\hat{r}_\pm}}$ for the iso-dual tesseract. Here, the exterior cube's 8 exterior iso-vertices in $V_{T^2_{\hat{r}_+}}$ are confined to the exterior IHR $T^2_{\hat{r}_+} \subset Y_+$ (not shown) in the exterior dynamical system while the interior cube's 8 interior iso-vertices in $V_{T^2_{\hat{r}_-}}$ are confined to the interior IHR $T^2_{\hat{r}_-} \subset Y_-$ (not shown) in the interior dynamical system, which are iso-dual to each other and are both iso-morphic, inter-locking, and synchronized to the initial cube [10].

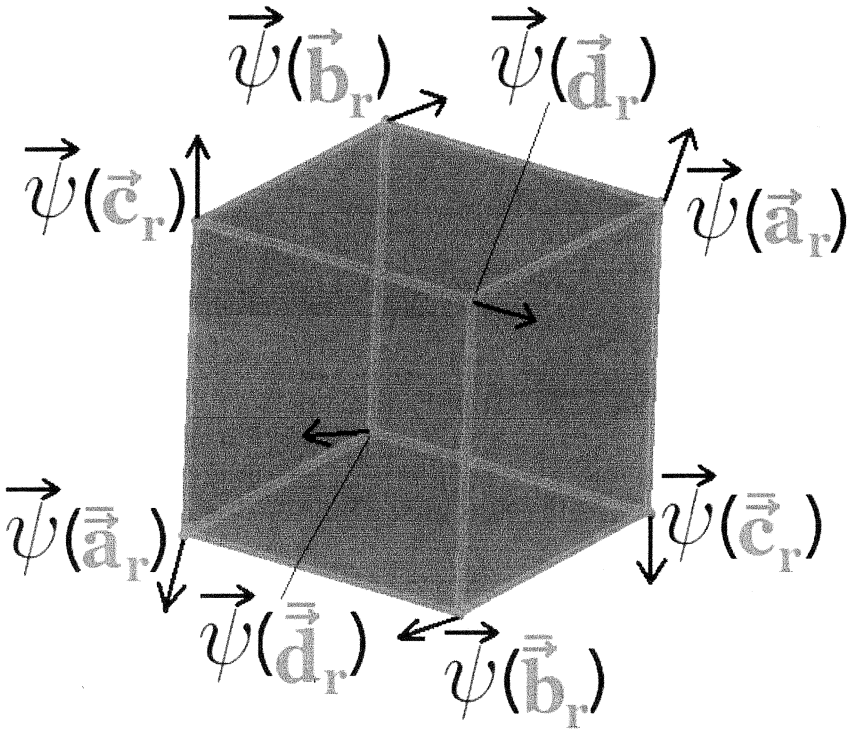

Fig. 4: The 8 triplex vertices of $V_{T_r^2} \subset T_r^2$ are assigned one layer of triplex order parameters [6, 7, 9] to encode topological deformations. These order parameter states can be iso-topically lifted [1, 2, 3, 4, 5, 6] to iso-triplex iso-vertex order parameter states in a double-projective fashion for the iso-dual tesseract.

8. Finally, we can simply select some positive-definite iso-unit with a corresponding inverse (i.e. we can reuse $\hat{r}_+$ and $\hat{r}_-$ or select alternative quantities) and repeat the iso-dual iso-topic lifting of Steps 1–6 for the vertice's triplex order parameters of eqs. (30–31) to define iso-triplex order parameters for the iso-dual tesseract. Thus, if we opt to redeploy $\hat{r}_+$ and $\hat{r}_-$ we can define the iso-dual iso-topic liftings

$$\begin{array}{rclcrcl}
\vec{\psi}(\vec{a}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{a}_r) \times \hat{r}_+ & | & \vec{\psi}(\vec{a}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{a}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{b}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{b}_r) \times \hat{r}_+ & | & \vec{\psi}(\vec{b}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{b}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{c}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{c}_r) \times \hat{r}_+ & | & \vec{\psi}(\vec{c}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{c}_r) \times \hat{r}_- \\
\vec{\psi}(\vec{d}_{\hat{r}_+}) & \equiv & \vec{\psi}(\vec{d}_r) \times \hat{r}_+ & | & \vec{\psi}(\vec{d}_{\hat{r}_-}) & \equiv & \vec{\psi}(\vec{d}_r) \times \hat{r}_- \\
\vec{\psi}(\bar{\vec{a}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\bar{\vec{a}}_r) \times \hat{r}_+ & | & \vec{\psi}(\bar{\vec{a}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\bar{\vec{a}}_r) \times \hat{r}_- \\
\vec{\psi}(\bar{\vec{b}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\bar{\vec{b}}_r) \times \hat{r}_+ & | & \vec{\psi}(\bar{\vec{b}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\bar{\vec{b}}_r) \times \hat{r}_- \\
\vec{\psi}(\bar{\vec{c}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\bar{\vec{c}}_r) \times \hat{r}_+ & | & \vec{\psi}(\bar{\vec{c}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\bar{\vec{c}}_r) \times \hat{r}_- \\
\vec{\psi}(\bar{\vec{d}}_{\hat{r}_+}) & \equiv & \vec{\psi}(\bar{\vec{d}}_r) \times \hat{r}_+ & | & \vec{\psi}(\bar{\vec{d}}_{\hat{r}_-}) & \equiv & \vec{\psi}(\bar{\vec{d}}_r) \times \hat{r}_-
\end{array} \tag{32}$$

for the double-projection iso-morphic transitions

$$\begin{array}{ccccc}
\vec{\psi}(\vec{a}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{a}_r) & \rightarrow & \vec{\psi}(\vec{a}_{\hat{r}_+}) \\
\vec{\psi}(\vec{b}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{b}_r) & \rightarrow & \vec{\psi}(\vec{b}_{\hat{r}_+}) \\
\vec{\psi}(\vec{c}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{c}_r) & \rightarrow & \vec{\psi}(\vec{c}_{\hat{r}_+}) \\
\vec{\psi}(\vec{d}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\vec{d}_r) & \rightarrow & \vec{\psi}(\vec{d}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{a}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\bar{\vec{a}}_r) & \rightarrow & \vec{\psi}(\bar{\vec{a}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{b}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\bar{\vec{b}}_r) & \rightarrow & \vec{\psi}(\bar{\vec{b}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{c}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\bar{\vec{c}}_r) & \rightarrow & \vec{\psi}(\bar{\vec{c}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{d}}_{\hat{r}_-}) & \leftarrow & \vec{\psi}(\bar{\vec{d}}_r) & \rightarrow & \vec{\psi}(\bar{\vec{d}}_{\hat{r}_+})
\end{array} \tag{33}$$

and the corresponding inverses

$$\begin{array}{ccccc}
\vec{\psi}(\vec{a}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{a}_r) & \leftarrow & \vec{\psi}(\vec{a}_{\hat{r}_+}) \\
\vec{\psi}(\vec{b}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{b}_r) & \leftarrow & \vec{\psi}(\vec{b}_{\hat{r}_+}) \\
\vec{\psi}(\vec{c}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{c}_r) & \leftarrow & \vec{\psi}(\vec{c}_{\hat{r}_+}) \\
\vec{\psi}(\vec{d}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\vec{d}_r) & \leftarrow & \vec{\psi}(\vec{d}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{a}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{a}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{a}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{b}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{b}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{b}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{c}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{c}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{c}}_{\hat{r}_+}) \\
\vec{\psi}(\bar{\vec{d}}_{\hat{r}_-}) & \rightarrow & \vec{\psi}(\bar{\vec{d}}_r) & \leftarrow & \vec{\psi}(\bar{\vec{d}}_{\hat{r}_+}).
\end{array} \tag{34}$$

At this point, we've completed the construction of the iso-dual tesseract by generalizing the dual 4D space-time IHR topology of Section 2.1 with the exterior and interior iso-duality [10].

3 Conclusion

In this research investigation, we deployed Santilli's iso-mathematics [1, 2, 3, 4, 5, 6] and Inopin's dual 4D space-time IHR topology [6, 7, 8, 9] as a platform to assemble the iso-dual tesseract from two inter-locking, iso-morphic, iso-dual cubes in Euclidean triplex space that fundamentally comply with exterior and interior IHR iso-duality [10]. To prove that such a tesseract can be built from one cube (rather than two distinct cubes), we presented the step-by-step procedure of Section 2 with simple, flexible, topologically-preserving instructions, where the single, initial cube was iso-topically lifted to simultaneously infer the exterior cube and the interior cube via double-projection. Subsequently, the exterior cube and the interior cube were inter-linked together in a point-by-point fashion by interconnecting the 8 iso-vertex pairs with 8 additional edges to superstruct the iso-dual tesseract. In total, the outcomes of this exploration are significant because an original iso-geometrical inter-connection between Santilli's exterior and interior dynamical systems has been established, which advances the application of iso-mathematics [1, 2, 3, 4, 5, 6] in a new direction.

We suggest that the next logical step of this research process should be to assign triplex order parameters [6, 7, 8, 9] to further encode topologi-

cal deformations and thereby define a complete "iso-dual tesseract wavefunction". From there, we may continue to launch from this platform to explore this frontier along various trajectories and assess the application of geno-mathematics and hyper-mathematics [1, 2, 3, 4, 5]. Thus, this developing iso-geometrical framework warrants further development, scrutiny, collaboration, and hard work in order to advance it for future application in the discipline of science.

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